THE
MATHEMATICAL GAZETTE

EDITED BY
T. A. A. BROADBENT, M.A.
WILLS HALL, BRISTOL, 9.

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ASYMPTOTES IN PARAMETRIC COORDINATES.

BY T. H. WARD HILL.

1. Let the curve be given by $x=f(t)$, $y=\phi(t)$.

2. *Asymptotes parallel to the coordinate axes.*

If as $t \to t_1$, $y=\phi(t) \to \infty$ while $x=f(t) \to a$, where a is finite, then $x=a$ is an asymptote.

If as $t \to t_2$, $x=f(t) \to \infty$ while $y=\phi(t) \to b$, where b is finite, then $y=b$ is an asymptote.

2.1. Consider the curve

$$x=(t+1)/(t-1), \quad y=(t+1)(t-1)/4t,$$

the parametric form of $y(x^2-1)=x$.

If $t \to 0$, $\quad y \to \infty \quad$ while $x \to -1$;

if $\quad t \to 1$, $\quad x \to \infty \quad$ while $y \to 0$;

if $\quad t \to \pm\infty$, $y \to \pm\infty$, while $x \to 1$.

Thus $x=\pm 1$ and $y=0$ are asymptotes of the curve.

2.2. To find on which side of the asymptote the curve lies.

Taking the curve of 2.1, for $t \to 0$ we have the asymptote $x+1=0$. Now for a point on the curve,

$$x+1=(t+1)/(t-1)+1=2t/(t-1).$$

So when $t>0$, that is, in the third quadrant, the curve lies on the left-hand side of the asymptote; for $t<0$, that is, in the second quadrant, the curve lies on the right-hand side of the asymptote.

Corresponding to $t \to 1$ we have the asymptote $y=0$. Now for a point on the curve,

$$y=(t+1)(t-1)/4t.$$

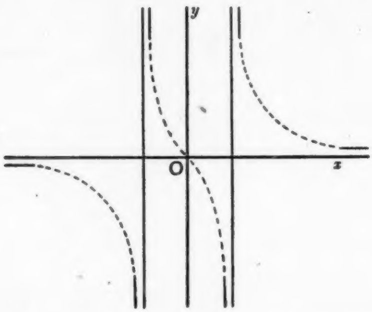
So when $t>1$, that is, in the first quadrant, the curve lies above the asymptote; for $t<1$, that is, in the third quadrant, the curve lies below the asymptote.

For $t \to \pm\infty$ we have the asymptote $x=1$. For a point on the curve

$$x-1=2/(t-1).$$

D

So when $t \to +\infty$, that is, in the first quadrant, the curve lies on the right-hand side of the asymptote, and when $t \to -\infty$, that is, in the fourth quadrant, the curve lies on the left-hand side of the asymptote.



It is understood that above, and in what follows, the phrase " $t > 0$, that is, in the third quadrant " means " for positive t in the vicinity of $t = 0$ ".

3. *Oblique asymptotes.*

Suppose now that as $t \to t_1$, both x and y tend to ∞. We shall use the definition that an asymptote is a straight line whose distance from a point on the curve tends to zero as the point moves along the curve to an infinite distance from the origin.

Thus, if $m = \tan\theta$ gives the direction of such a line, and if the axes are rotated through an angle θ, one of the asymptotes of the curve will be parallel to the new x-axis. So the new ordinate $-x\sin\theta + y\cos\theta$ will tend to a finite limit as x and y tend to ∞, that is,

$$(-x\sin\theta + y\cos\theta)/x \to 0,$$

that is,

$$-\sin\theta + (y/x)\cos\theta \to 0.$$

So we have

$$m = \tan\theta = \lim_{t \to t_1} \frac{y}{x} = \lim_{t \to t_1} \frac{\phi(t)}{f(t)},$$

where t_1 is such that $f(t)$ and $\phi(t) \to \infty$ as $t \to t_1$.

3.1. We cannot have $\sin\theta = 0$ or $\cos\theta = 0$, for then $-x\sin\theta + y\cos\theta$ could not tend to a finite limit. It is otherwise obvious that rotation would be unnecessary.

3.2. If m tends to a finite limit, this gives us an " asymptotic direction " for the curve. If in addition

$$y - mx\ [= (-x\sin\theta + y\cos\theta)/\cos\theta]$$

tends to a finite limit, c say, as $t \to t_1$, the corresponding asymptote is

$$y = mx + c.$$

The method of obtaining the position of the curve with respect to an asymptote is best illustrated by an example.

3.3. Take the curve

$$x=6t^2/(t-1)(2t-1), \quad y=2t(2t^2+1)/(t-1)(2t-1)$$

the parametric form of $(y-x)^2x-3y(y-x)+2x=0$.

For $t \to \pm\infty$, $y \to \pm\infty$ and $x \to 3$. Thus $x=3$ is an asymptote, and the position of the curve with respect to this asymptote is obtained as in 2.2.

For $t \to 1$, $x \to \infty$ and $y \to \infty$;

$$m = \lim_{t\to 1} \frac{y}{x} = \lim_{t\to 1} \frac{2t(2t^2+1)}{6t^2} = 1\ ;$$

also

$$\begin{aligned}\lim_{t\to 1}(y-x) &= \lim_{t\to 1} \frac{2t(2t^2+1)-6t^2}{(t-1)(2t-1)} \\ &= \lim_{t\to 1} \frac{2t(2t^2-3t+1)}{(t-1)(2t-1)} \\ &= \lim_{t\to 1} 2t \\ &= 2.\end{aligned}$$

So $y=x+2$ is an asymptote. Now for a point on the curve,

$$y-x-2=2t-2=2(t-1).$$

Hence when $t>1$, that is, in the first quadrant, the curve lies above the asymptote, and when $t<1$, that is, in the third quadrant, the curve lies below the asymptote.

For $2t-1 \to 0$, $x \to \infty$ and $y \to \infty$ and

$$\begin{aligned}m_1 &= \lim_{t\to \frac{1}{2}} \frac{y}{x} \\ &= \lim_{t\to \frac{1}{2}} \frac{2t(2t^2+1)}{6t^2} \\ &= 1,\end{aligned}$$

and

$$\lim_{t\to \frac{1}{2}}(y-x) = \lim_{t\to \frac{1}{2}} 2t = 1.$$

Hence $y=x+1$ is an asymptote; the position of the curve with respect to this asymptote is found as above.

4. The equations to tangents to a curve, and the position of the curve with respect to the tangent at any point, may be found by using the same methods.

5. *Parabolic branches.*

The equation to the parabola $y^2=4ax$ may be written $y/x=4a/y$. Thus, as x and y both tend to infinity, $y/x \to 0$. This would appear to give Ox as an asymptotic direction; yet there is no rectilinear asymptote in this direction.

5.1. Now suppose that as $t \to t_1$, $x=f(t) \to \infty$, $y=\phi(t) \to \infty$ and

$$\lim_{t\to t_1} y/x = 0.$$

By analogy with the parabola the curve is said to have a *parabolic branch* in the asymptotic direction Ox.

Similarly if for $t \to t_2$, $x=f(t) \to \infty$, $y=\phi(t) \to \infty$ and

$$\lim_{t\to t_2} y/x = \infty,$$

the curve has a parabolic branch in the asymptotic direction Oy.

5.2. We have now to investigate the case

$$\lim y/x = m, \quad \lim (y - mx) = \infty,$$

when both x and y tend to infinity as $t \to t_1$ and m is finite.

$\lim y/x = m$ gives us an asymptotic direction, say θ, for the curve. If the coordinate axes are rotated through an angle θ, the curve will have an asymptotic direction in the direction of the new x-axis, and the case reduces to one of those considered in 5.1.

We shall have, then, a parabolic branch in the direction given by θ. In the following example it is shown how successive approximations to the form of such a parabolic branch may be obtained.

5.3. Consider the curve

$$x = 2a/(t+1)(t-1)^2, \quad y = 2at/(t+1)(t-1)^2,$$

the parametric form of $(y-x)^2(y+x) = 2ax^2$.

Here $m = y/x = t$.

Both x and y tend to infinity as $t \to \pm 1$.

For $t \to -1$, we have, as in 3.3, the rectilinear asymptote $y + x = \frac{1}{2}a$, the curve lying above the asymptote in the fourth quadrant and below it on the second quadrant. For $t \to 1$, $m = 1$ and

$$y - x = 2a/(t+1)(t-1) \to \infty \quad \text{as } t \to 1.$$

So there is a parabolic branch in the direction $y = x$. To find an approximation to the shape of the curve "at infinity", put $t = 1 + h$, so that when $t \to 1$, $h \to 0$, and we have $x = 2a/(2+h)h^2$, $y = 2a(1+h)/(2+h)h^2$.

First approximation.

$$x = a/h^2(1 + \tfrac{1}{2}h) \qquad = a(1 - \tfrac{1}{2}h)/h^2,$$

$$y = a(1+h)/h^2(1+\tfrac{1}{2}h) = a(1+h)(1-\tfrac{1}{2}h)/h^2$$
$$= a(1 + \tfrac{1}{2}h)/h^2.$$

Thus $$y + x = 2a/h^2, \quad y - x = a/h,$$

whence $$(y-x)^2 = \tfrac{1}{2}a(y+x).$$

Second approximation.

$$x = a(1 - \tfrac{1}{2}h + \tfrac{1}{4}h^2)/h^2,$$
$$y = a(1+h)(1 - \tfrac{1}{2}h + \tfrac{1}{4}h^2)/h^2$$
$$= a(1 + \tfrac{1}{2}h - \tfrac{1}{4}h^2)/h^2.$$

Thus $$y + x = 2a/h^2, \quad y - x + \tfrac{1}{4}a = a/h,$$

whence $$(y - x + \tfrac{1}{4}a)^2 = \tfrac{1}{2}a(y+x).$$

If y_C, y_P are the ordinates of points on the curve and parabola respectively, corresponding to the same abscissa, we have, using the second approximation to the value of y_P,

$$y_C - y_P = \frac{a}{h^2}\left[\frac{2(1+h)}{2+h} - (1 + \tfrac{1}{2}h - \tfrac{1}{4}h^2)\right] = ah/4(2+h).$$

So when $h > 0$, that is, in the first quadrant (the upper branch of the curve), the curve lies above the parabola; and when $h < 0$, that is, in the first quadrant (the lower branch of the curve), the curve lies below the parabola. In both cases, the branches approach the parabola from the outer sides.

T. H. W. H.

THE CELESTIAL CYLINDER.

By Sir Percy Nunn.

§ 1. This article offers suggestions, not for formal lessons on astronomy, but for occasional "raids" into astronomical territory—excursions intended to show how even an elementary knowledge of mathematics may be applied fruitfully to problems of real importance and abiding interest. The mathematical knowledge assumed is acquaintance with the three main trigonometrical ratios and their reciprocals, familiarity with the "natural" trigonometrical tables, and a little practice in changing the subject of a straightforward formula containing the ratios. Values taken to three places generally suffice, and skill in using logarithms, though desirable, is not demanded.

I

§ 2. The sun, as all know, moves daily across the sky from east to west and at night harks back beneath the horizon to start again next day. A little observation during the hours of darkness shows that the stars also move across the sky in the same way, and in fact revolve daily about an axis directed nearly, but not precisely, to the star called, for that reason, the Pole Star. Any boy or girl may easily follow the nightly movement of the Plough,* the well-known star-group which, in northern latitudes, is always above the horizon. A sufficiently determined observer, making the first record of its position at (say) 6 p.m. on a winter's evening and the last at 6 a.m. next morning, will see plainly that during this period the Plough has gone about half-way round the sky.

At the north pole the Pole Star is practically in the zenith ; on the equator (as those who cross the Line may often see) it is practically on the horizon. If one travels pole-wards from the equator it rises in proportion to the arc traversed ; that is, the altitude of the pole is everywhere equal to the latitude. This fact explains why, in different latitudes, the gnomons of sundials are differently inclined : their edges must always point towards the pole.

§ 3. *The Celestial Cylinder.* In studying the places and motions of the stars it has been usual to regard them as if they were (as Lorenzo said to his Jessica) "patines of bright gold" fixed in a sphere which rotates diurnally about the polar axis. For elementary study there is, however, much to be said for thinking of them as "inlaid", not in a sphere but in a cylinder. For, in the first place, it needs an expert to make a globe, while anyone can, in a trice, make a cylinder out of a rectangle of drawing-paper. And in the second place, since a cylindrical surface can be spread out flat, it presents problems of plane, not of spherical trigonometry. There is the drawback that only a cylinder of infinite length could accommodate all the stars ; but to neutralise this defect, those around the poles may be shown, if need be, on discs that fit the top and bottom of the cylinder.

Let us then proceed to imagine the making of a "celestial cylinder". If one is actually made for exhibition to a class, it would be wise, for ease of calculation, to let its radius be 10 cm. A height of 60 cm. is sufficient for most purposes. Thus the cylinder, when flattened out, would become a rectangle measuring $2\pi \times 10 = 62{\cdot}8$ cm. by 60 cm. This is to be covered with a rectangular network of lines. The first line will be VV' (Fig. 1)—plainly the trace of a plane, the "equatorial plane", which divides the whole sky into

* Star-gazers in the southern hemisphere have no Pole Star, but their Southern Cross is a magnificent substitute for the Plough. In reading this article they must regularly substitute "south" for "north" and "north" for "south".

a northern and a southern half. It passes through the observer's eye, is perpendicular to the polar axis, and cuts the horizon in the east and west points. It may be supposed to trace across the sky an arch called the "celestial equator", whose highest point lies on the meridian directly above the south point in the northern hemisphere and the north point in the southern hemisphere. Let the latitude of the place of observation be λ. Then since the inclination of the polar axis to the horizontal plane is λ, the inclination of the equatorial plane must be $90° - \lambda$, and this must also be the altitude of its highest point. It will often be convenient to represent $90° - \lambda$ by the symbol λ'.

§ 4. *Declination.* When the rectangle is bent into a cylinder, the equator, VV', becomes a circle from whose centre, O, the eye is supposed to view the heavens. If, being at O, you raised your gaze from the plane of the celestial equator through an angle β, your line of sight would meet the cylinder in a point whose height above the equator is $r \tan \beta$, r being the radius of the cylinder. The angle β is called the "declination" of the point, and the circle drawn through it around the cylinder may conveniently be called a "declination-parallel". Declination-parallels obviously correspond to parallels of latitude.

On the rectangle a declination-parallel becomes, of course, a line parallel to VV' and distant $r \tan \beta$ above or below it. We shall suppose that lines representing declination-parallels have been drawn for declinations positive and negative (that is, above and below the equator), rising by steps of 5° to 70°. Since $30 = 10 \tan 71° 33'$, one can go no farther; stars with declination beyond 71° 33′ must find a place on the discs which close the ends of the cylinder. On these a declination-parallel is a circle whose radius (from the pole) is $10 \tan \beta'$, where $\beta' = 90° - \beta$.

Lines corresponding to meridians of longitude cross VV' at right angles. On the cylinder they are traces of planes which all contain the polar axis and may be thought to mark out arches running across the sky from pole to pole. They will be drawn on the rectangle at intervals corresponding to 15°, *i.e.* 62·8/24 cm.

These lines may be looked at from two points of view. From the first they are stationary lines—"hour-lines"—across which the stars travel from east to west along the parallels of declination. For instance, we learn from the list of fixed stars in the invaluable pages of *Whitaker's Almanack* * that the declination of the brilliant star, Sirius, is −16° 38′. Its track across the rectangle will accordingly be a declination-line $10 \tan 16° 38' = 3$ cm. below VV'. In the course of every hour it will move from one of the hour-lines to the next. The zero hour-line is the meridian, the middle one of the series, running through the south point of the horizon. The others are numbered from it to right and left. Thus a star on the nth hour-line to the left has still to go n hours before it reaches the meridian; one on the nth line to the right crossed the meridian n hours ago.† In both cases its "hour-angle" (negative or positive) will be n hours or $15\,n°$. Thus the statement of a star's hour-angle and declination is a statement of its position at a given time with reference to the meridian and the equator.

* "Whitaker" is indispensable, but need not be the current issue. The official *Nautical Almanac* (H.M. Stationery Office) is a desirable luxury. The issue for 1938 includes the late Professor Fotheringham's masterly article on the Calendar, but much of this matter is in *Whitaker*.

† It is assumed that the rectangle represents the cylinder as seen from the point O within it. If the appearance from without is needed, "right" and "left" must be interchanged.

From the second point of view the lines in question are movable; they turn round with the cylinder, and the stars may be thought of as attached to them. We shall now call them "lines of right ascension (R.A.)". The statement of a star's R.A. and declination serves to fix its position with reference to the other stars. Any suitable star might have been chosen arbitrarily to mark the line of zero right ascension, but as a matter of fact, this line is determined by the movements of the sun. Twice a year the sun crosses the equator—on his way north at the vernal equinox (about 21 March) and again on his way south at the autumnal equinox (about 23 September). The line of zero R.A. is the one which passes through the former crossing-point—known as "the first point of Aries". The other lines follow from west to east, the way the sun travels among the stars.* They are numbered in hours from 1 h. to 23 h.

§ 5. *Sidereal Time.* Now imagine an observer to shrink in size (like the famous Alice) until he can ensconce himself comfortably at O within the cylinder, and let the cylinder turn about its axis at such a rate that a star which appears, at any moment, attached to a particular R.A. line shall remain attached to that line as the cylinder revolves. This arrangement will, of course, require a special piece of clockwork, and such clockwork exists, in effect, in every astronomical observatory. Since it keeps step with the stars it is said to record "sidereal" time.

A sidereal clock should register zero when the first point of Aries is on the meridian. Let us suppose, then, that (as is possible) the sun occupied that point—*i.e.* that it crossed the equator—at the moment on 21 March when it was also on the meridian.† Then at that moment the sidereal clock should read zero. It would read zero again when the first point of Aries returned to the meridian next day, but the sun would not also be there. For during the interval it would have travelled about 1° eastwards among the stars, and would so have fallen about 4 minutes behind in the daily procession towards the meridian. Thus the sidereal day is about 4 minutes shorter than the solar day—actually on the average 3 m. 56 s. shorter. *Whitaker* records for each day of the year what the sidereal clock should read when the local mean time is 12 o'clock midnight, *i.e.* at the legal beginning of the day.‡

It will now be clear that to determine the R.A. of any star it is necessary only to note (with a "transit circle") the sidereal time at which it crosses the meridian; for that time *is* its R.A. The same observation can also be used to fix its declination; for the declination is simply the angular distance of the star, up or down the meridian, from the point where the equator

* Here is a simple proof that the sun does so travel. If a prominent constellation (*e.g.* Orion, which begins, in Britain, to appear in December) be watched during two or three months, it will be seen to reach the same place in the sky rather less than half an hour earlier every week. Now our clocks keep pace with the "mean" sun; hence the sun must be approaching Orion at a rate which would carry it round the heavens in something over 48 weeks—actually in about 365¼ days.

† Normally it occupies the point some time before or after reaching the meridian on the day of the equinox.

‡ At places on the central meridian of a "time zone", *i.e.* in longitude 0°, 15°, 30°, 45°, etc., east or west of Greenwich, local mean noon is just 12 o'clock. At places east or west of the central meridian it occurs earlier or later than 12 o'clock, the difference being 4 min. for every degree of longitude. For instance, mean noon occurs at Bristol (2½° west of Greenwich) at 12.10 p.m., at Stockholm (3° east of the 15th meridian) at 11.48 a.m. by the standard clocks of the city.

The record in *Whitaker*, taken with the preceding paragraph, makes it easy to train a clock to gain 4 minutes a day and to keep sidereal time. Such a clock is useful as a question-provoking object in a mathematical class-room or a laboratory.

crosses it. And that point, we have seen, is λ', *i.e.* $90° - \lambda$ above the horizon. For instance, if, in latitude 42° N., a star crosses the meridian at an altitude of 62°, its declination must be $62° - 48° = +14°$. If the transit-altitude were 37°, the declination would be $37° - 48° = -11°$.

§ 6. *Latitude at Sea.* Conversely, when the transit-altitude of a star is known it is easy to find λ' and so to fix the latitude. The " star " used for the purpose at sea is nearly always the sun. *Example.* A sea-captain observing the sun towards noon on 20 May 1941, determined that its transit-altitude was 57°. Now on that day, by *Whitaker*, the sun's declination was $+20°$, *i.e.* it was, at transit, 20° above the equator. Hence the altitude of the equator was $57° - 20° = 37°$. This is the value of λ' or $90° - \lambda$; hence

$$\lambda = 90° - 37° = 53° \text{ N.}$$

If the date had been 13 November, when the sun's declination was $-18°$, the co-latitude λ' would have been $57° + 18° = 75°$, and the latitude would have been 15° N.

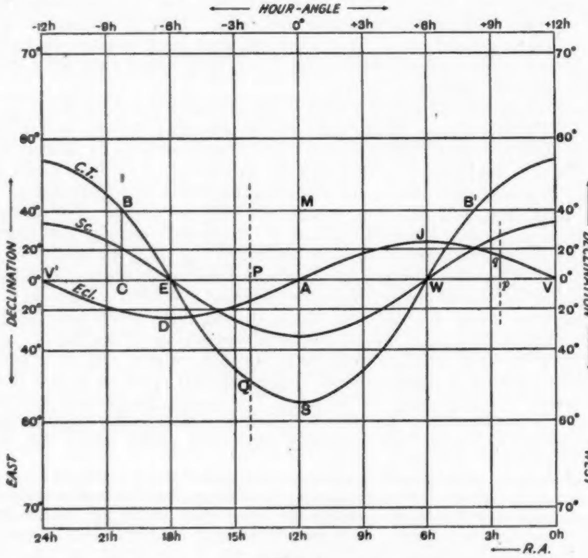


FIG. 1.

§ 7. *Longitude at Sea.* Observation of the *time* of transit by a clock keeping Greenwich time tells the sea-captain the time (G.M.T.) of local noon, and thus discloses his longitude. But an important correction comes in here. The poet Hood's declaration that " the sun will never fail " gives it praise sadly undeserved. It may arrive on the meridian more than 16 minutes before its time in November, and more than 14 minutes after in February. *Whitaker* records for every day the " equation of time "—that is, the correction needed to make the noon actually observed agree with the time at which it would occur if the sun always moved at its average rate.

Example. On 13 November the sun was south at 15 h. 32 m. by a clock keeping G.M.T. by means of wireless signals. On that date the equation of time was nearly + 16 m. Hence the time of the local mean noon (as distinguished from the observed noon) was 11 h. 44 m. This was the local mean time when the G.M.T. was 15 h. 32 m., so that the ship's longitude was

$$15 \text{ h. } 32 \text{ m.} - 11 \text{ h. } 44 \text{ m.} = 3 \text{ h. } 48 \text{ m. or } 57° \text{ W.}$$

II

§ 8. *The Horizon.* Imagine the cylinder to be mounted with its axis duly pointing to the pole, and rotating at the sidereal rate. Let a horizontal plane pass through its central point O. Then this is the plane of the horizon, and its trace upon the cylinder may be called an " horizon-curve ". At any given moment (at night) all stars whose places are above that curve will be visible from O; all the rest will be invisible. In Fig. 1 the curves marked " Sc." and " C.T. " are traces of the horizontal plane corresponding to two different inclinations of the polar axis, *i.e.* to two different latitudes.

Fig. 2, representing a fragment of the cylinder viewed from without, shows how an horizon-curve may be plotted upon the rectangle. EW is the east-to-west line through O, $EPAW$ is one-half of the equator, $EQSW$ the lower half of the curve " C.T." in Fig. 1. Since the inclination of the equatorial to the horizontal plane is λ', the inclination of the horizontal to the equatorial, which we have to deal with here, is $-\lambda'$. AS and PQ are segments of the hour-lines with the same lettering in Fig. 1. The triangles AOS and PNQ

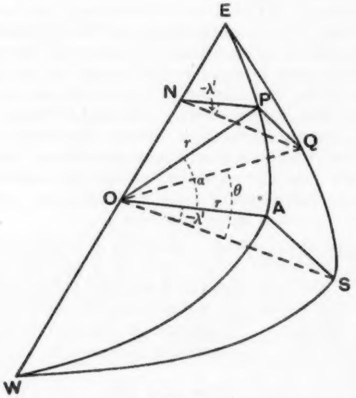


FIG. 2.

are sections of the figure perpendicular to EW. It follows that AO, SO, PN and QN are all at right angles to EW. Since the equatorial plane $EPAW$ is perpendicular to the axis, and AS and PQ are parallel to it, it follows that OAS and NPQ are also right angles. The angles AOS and PNQ, measuring the angle between the planes, are both $-\lambda'$. Lastly $\angle AOP = \alpha$, measured along the equator from A, is the hour-angle of the hour-line PQ (cf. Fig. 1).

Now in the right-angled triangle PNQ, $PQ = PN \tan(-\lambda')$; and in the right-angled triangle PNO, $PN = OP \sin PON = r \cos \alpha$,

whence $$PQ = r \cos \alpha \tan(-\lambda') \quad \text{.............................(1')}$$

and $$PQ = -r \cos \alpha \cot \lambda. \quad \text{.................................(1)}$$

Putting $r = 10$ and choosing any latitude λ, one can in a few minutes calculate the ordinates for the relevant horizon-curve to be added to Fig. 1. The one marked " Sc.", calculated for lat. 56°, would do fairly well for Edinburgh and Glasgow, and also for Stockholm and Moscow. The one marked " C.T." is for lat. 34°, and would serve roughly for Capetown, Adelaide, Sydney, Auckland and Buenos Ayres in the southern hemisphere, together with Washington and Los Angeles in the northern.

In formula (1), if $\alpha = \pm 90°$, $PQ = 0$ for all latitudes. That means that all horizon-curves pass through the east and west points, E and W. Where $\alpha = 0°$, $PQ = AS = -r \cot \lambda$. Where $\lambda = 90°$ (*i.e.* at the poles) this gives $AS = 0$; that is, the horizon and the equator coincide. At the equator, where $\lambda = 0°$, AS is infinite, and the curve becomes a pair of lines through E and W at right angles to VV'.

Imagine a star to be carried by its daily rotation along the declination-parallel, say of 40°. Coming from the left-hand edge of the rectangle, it will be seen to rise in latitude 34° as soon as it crosses the horizon-curve at B, to be south when it reaches M, and to set when it crosses the curve again at B'. The length BB', measured on the scale of hour-angles, gives its total time above the horizon; it appears to be about $16\frac{1}{2}$ hours.

The declination-parallel for 40° lies wholly above the curve for 56°. In that latitude, therefore, the star will neither rise nor set, but will be, like the Plough in Britain, circumpolar.

If a star's declination is 20° N. (this is the sun's declination about 20 May and again about 23 July) it will cross the curve " Sc." before reaching the curve " C.T.", and cross the latter again before crossing the former. It follows that on the dates named the day is a good deal longer in lat. 56° than in lat. 34°. But if the declination is 20° S. (this is the sun's declination about 21 November and again about 21 January) the reverse is true. These remarks apply also to the longest and the shortest days, when the sun's declination is $23\frac{1}{2}°$. The curves show that there is much less difference between them in low latitudes than in high. At the equator, where the horizon-curve is a pair of parallel lines, all days are equally long and all stars are visible for twelve hours. At the poles the stars never rise nor set.

§ 9. *Hours of Rising and Setting.* A given star will be south at the same *sidereal* time all over the world; for this time is just the star's R.A. The only qualification is that if the R.A. changes sensibly during twenty-four hours (as the sun's does, and still more the moon's) an allowance must be made based upon the observer's longitude. A glance at *Whitaker's* tables will show what the allowance amounts to. For the sun it amounts to only about 1 minute for 90° of longitude.

The rising and setting occur before and after southing at an interval which (as the curves show) depends on the latitude. It can be calculated by means of (1). In Fig. 1 let β be the declination of the line which crosses a horizon-curve at B. Then the ordinate $BC = r \tan \beta$. This may be taken to be the value of PQ in (1), whence we obtain:

$$r \tan \beta = -r \cos \alpha \cot \lambda,$$

or $$\cos \alpha = -\tan \beta \tan \lambda. \quad \text{...............................(2)}$$

This formula gives (in degrees) the hour-angle for both rising and setting. If the earth were airless, like the moon, it would give the exact value of α. But the effect of the earth's atmosphere upon stars near the horizon is the

same as the effect of the water in a swimming bath upon objects lying on the bottom: they seem to be raised above their actual positions. In this way the stars are visible unless they are more than 34′ below the horizon. Hence the time during which a star can be seen is always rather larger than the time given by (2).

We shall see later how the correction for this "refraction" is made. At present we take the results from (2) to find the approximate times of rising and setting. *Example.* At what time does Sirius rise and set on 1 January (*a*) at Edinburgh and (*b*) at Capetown?

(*a*) By *Whitaker* the R.A. of Sirius is 6 h. 42 m., and the sidereal time at noon on 1 January is 18 h. 40 m. Sirius, then, will be south at sidereal time 6 h. 42 m., that is (24 h. − 18 h. 40 m.) + 6 h. 42 m. = 12 h. 2 m. after midday. But the interval is measured in sidereal time, which gets 4 minutes ahead of solar time in 24 hours. To obtain the local mean time of Sirius's southing we ought, then, to subtract 2 minutes, making the time 12 h. 0 m. after midday, *i.e.* midnight.

Now the declination of Sirius is −16° 38′, so we have for Edinburgh:

$$\cos \alpha = -\tan 56^\circ \tan(-16^\circ 38')$$
$$= \cos(\pm 63^\circ 43'),$$
$$\alpha = \pm 63^\circ 43' \text{ or } \pm 4 \text{ h. } 15 \text{ m.}$$

As before, the hour-angle is computed in sidereal time. Reckoning it as one-sixth of a day we must subtract one-sixth of 4 min. Thus we infer that the times of rising and setting are, by local mean time, about 12 h. ±4 h. 14 m.; that is, that Sirius rises at 7.46 p.m. on 1 January and sets at 4.14 a.m. on 2 January.

Finally, since Edinburgh clocks keep G.M.T., but the town's longitude is 3° 11′ W., the local time is nearly 13 minutes behind the standard time. We conclude, therefore, that by G.M.T. Sirius rises at about 7.59 p.m. and sets at about 4.27 a.m.

(*b*) Since Capetown is in the southern hemisphere, "south" in the previous problem must be replaced by "north". The hour-angle α must be calculated from the time when the sun is north, and in applying (2) the negative latitude and declination should be treated as positive. Thus:

$$\cos \alpha = -\tan 34^\circ \tan 16^\circ 38'$$
$$= \cos(\pm 101^\circ 40').$$

The time-equivalent is about 6 h. 47 m., or in solar units about 6 h. 46 m. Sirius will be north at 6 h. 42 m. + 12 h. = 18 h. 42 m., sidereal time, and this is, on 1 January, 2 minutes after noon by local mean time. Capetown keeps the time of the meridian of 15° E., which is one hour fast on Greenwich, but since the city is 3½° farther east its local time is 14 minutes ahead of the standard time. We conclude, then, that at Capetown on 1 January Sirius will rise and set at about 11 h. 48 m. ±6 h. 46 m. by standard time.

§ 10. *The Place of Rising and Setting.* Returning to the hypothetical star which, in Fig. 1, rose above the horizon "C.T." when it reached B, we now ask *where* the rising took place. The answer needed is the angular distance from the point S along the horizon to B. That distance is called the "azimuth", and Fig. 2 shows how to calculate it. It will be measured from the south point in the northern hemisphere and from the north point in the southern hemisphere.

The azimuth is $\angle SOQ = \theta$, and, since SO and QN are parallel, it is equal to $\angle OQN$. Thus in the right-angled triangle QNO:

$$\cot \angle OQN = \cot \theta = \frac{NQ}{ON}.$$

Now in the right-angled triangle QPN

$$NQ = NP \sec(-\lambda') = NP \operatorname{cosec} \lambda,$$

and in the right-angled triangle PNO

$$NP = r \cos \alpha \text{ and } ON = r \sin \alpha.$$

Hence $\cot \theta = (r \cos \alpha \operatorname{cosec} \lambda)/r \sin \alpha = \cot \alpha \operatorname{cosec} \lambda,$

or $$\tan \theta = \tan \alpha \sin \lambda. \qquad (3)$$

By this formula the azimuth of the points of rising and setting can be found when the hour-angle is known.

Example. Where does Sirius rise and set at Capetown on 1 January?

In § 9 we saw that $\alpha = 101° 40'$; so we have

$$\tan \theta = \tan 101° 40' \sin 34°$$
$$= \tan 110° 16';$$

that is, the rising and setting occur at points on the horizon 110° 16′ east and west of the north point.

III

§ 11. *The Ecliptic.* The path the sun appears to trace annually against the background of the fixed stars is called the "ecliptic". In using Fig. 1 to study it we must now regard the lines perpendicular to VV' as lines of R.A., and the horizon-curves, being irrelevant, must be wholly disregarded. *Whitaker* gives the sun's R.A. and declination for every day, and a selection of the entries can be used to trace the ecliptic upon the rectangle. Starting from V (the first point of Aries and the vernal equinox) it rises to the June solstice, J, where the R.A. is 6 h. and the declination $+23° 27'$, descends to cross the equator again at the autumnal equinox, A, (R.A. 12 h., decl. zero), continues the downward path to the December solstice, D (R.A. 18 h., decl. $-23° 27'$), and once more ascends to cross the equator at V'—a point which is identical, of course, with V.

Does the trace of the ecliptic around the cylinder lie, like the equator, in a plane? If so, its inclination, i, to the plane of the equator must be 23° 27′, and its intersection with that plane must run from V to A through O, the central point of the cylinder; moreover, the ordinates of the curve in Fig. 1 must conform with formula (1′), suitably modified. First there is the obvious substitution of $i = +23° 27'$ for $-\lambda'$. In the second place we must, in calculating (say) the ordinate pq, use the angle Wp just as in (1′) we used the angle AP. Now Wp is 6 h. (or 90°) *minus* pV, and pV is the R.A. of q. Thus if we now take α to represent the R.A. in degrees we must replace α in (1′) by $90° - \alpha$. So the formula becomes

$$pq = r \sin \alpha \tan i.$$

But in Fig. 1 $pq = r \tan \beta$, where β is the sun's declination. We conclude, then, that, if the ecliptic lies in a plane, the value of pq corresponding to a given value, α, of the sun's R.A. should be $r \tan \beta$, or

$$\sin \alpha \tan i = \tan \beta. \qquad (4)$$

Example. On 1 May 1940 the sun's R.A. (to the nearest minute) was 2 h. 34 m. or 38° 30′. Substituting in (4) we have

$$\sin 38° 30' \tan 23° 27' = \tan 15° 6{\cdot}7'.$$

Whitaker gives the declination as 15° 6·9′. Since we have worked only to the nearest minute of time, the agreement is sufficiently close. A few more such computations, made for dates taken at random and yielding the same degree of conformity with the data, will leave no doubt that the familiar phrase " the plane of the ecliptic " describes a fact.

§ 12. *Celestial Longitude*. Since the sun is always somewhere on the ecliptic only one measurement or coordinate is needed to fix its position. Suppose it to be at q in Fig. 1, turn the rectangle into the cylinder, place the observer at O, and let his line of sight sweep along the plane of the ecliptic from V, the first point of Aries, to q. Then the angle through which it has swept is the needed coordinate. It is called the sun's longitude, and is measured in degrees.

The problem of calculating the position of q on the ecliptic, given the R.A., corresponds with the problem of finding the position of B on the horizon-curve, given the hour-angle. In (3) θ stands for the angle SOB measured along the horizon from S to B. In the present problem the corresponding angle is VOq measured along the ecliptic. Since J lies one-quarter of the way round the ecliptic, its longitude must be 90° ; hence if θ now stands for the longitude of q, the angle θ in (3) must be replaced by $90° - \theta$. Also i will be put instead of $-(90° - \lambda)$, or $i + 90°$ instead of λ ; and as before $90° - \alpha$ will replace the α of (3). So we have

$$\tan(90° - \theta) = \tan(90° - \alpha)\sin(i + 90°),$$

$$\cot\theta = \cot\alpha\cos i,$$

or

$$\tan\theta = \tan\alpha\sec i. \quad\text{(5)}$$

Example. The sun's R.A. on 4 November may be taken to be 14 h. 34 m. or 218° 30′. What is its longitude? Since 218° 30′ = 180° + 38° 30′, we have

$$\tan\theta = \tan 38° 30' \sec 23° 27'$$
$$= \tan(180° + 40° 56'),$$

– the 180° being inserted to conform with the facts as shown in Fig. 1. Hence the sun's longitude is 220° 56′.

IV

§ 13. *Cylindrical Star Maps*. The altitude of a star upon crossing the meridian depends, we know, upon its declination and the latitude—being equal to the co-latitude *plus* or *minus* the declination. Moreover we know that the star will be on the meridian at the same sidereal time all over the world. Its position at other times depends upon the latitude, the declination and the hour-angle, and is the same at the same sidereal time at all places which have the same latitude. These facts suggest the construction of star-maps to show the position of the stars in a particular latitude at a particular sidereal time. In making such maps we may again have resource to the cylinder, but it must now be a cylinder with a vertical axis set up upon the plane of the horizon. It would be awkward for a watcher of skies actually to hold such a cylinder around his head ; so we shall suppose each map to be composed of two sheets—one for the southern, the other for the northern sky. In use they would be bent into the semi-cylindrical form or supposed to be so bent. The division of the cylinder into sheets would be made along the verticals through E and W of Fig. 1, and the two halves of the meridian would be the central lines of the sheets. A scale of azimuths would be recorded along the base of each sheet—running from 0° to ±90° each way on the southern, and from 180° to ±90° each way on the northern sheet. The edges of the sheet

would be graduated for altitude on a tangent-scale. That is, the lines of equal azimuth and equal altitude (if they were drawn) would be exactly like the lines crossing the upper half of Fig. 1. It would be unnecessary and confusing actually to draw the lines, but scales drawn along the margins would be essential.*

All stars with the same R.A. lie on the same hour-line, and all stars with the same declination follow the same tracks across the sky. The lines actually to be drawn on the maps are, accordingly, a set of hour-lines crossed by a set of declination-lines. Suppose such a map to be made for use in latitude λ at (say) 8 hours sidereal time. Then on the southern sheet the central-line will be marked "8 h.", and the others will be marked in due order from right to left. In other words, the hour-lines will be converted into R.A. lines adjusted to the assumed time of observation. On the northern sheet the central line will also be marked "8 h." down to the point representing the pole (at altitude λ), but the lower part will be marked "20 h.". The other R.A. lines will continue those of the southern sheet across the eastern and western edges, and will bear the same numbers.

The shapes of the hour-lines will be the same in every map drawn for the same latitude; only the numbering which turns them into R.A. lines will differ in accordance with the sidereal time assumed. The declination-lines will be in all respects the same. Thus when the lines have been drawn for a pair of sheets they may be reproduced as often as is desired by pricking holes through the line-crossings or by some more elaborate method.

The question how to plot the lines now arises. It involves calculating the azimuth and altitude of all the points at which they cross; when these calculations are made the lines are easily drawn.

§ 14. *The Formulae.* A study of Figs. 3 and 4 supplies the formulae needed to determine the azimuth and altitude of a crossing-point when its R.A. and declination are stated. Fig. 3 is a section of the celestial cylinder through the polar axis OA and the meridian BS. NS is the trace of the horizon-plane, N being the northern and S the southern point. OM is one-half of the trace of the equator-plane, so that $\angle AON=\lambda$ and $\angle MOS=\lambda'$. The line OB is directed towards a star, now on the meridian, whose declination, $\angle MOB$, is β. As time goes on OB will trace out a cone whose semi-angle DOB is β', and B will travel along the declination-parallel whose diameter is BC. The dotted circle represents this parallel, twisted for study through a right-angle about BC. P is the position the star has reached on it when the hour-angle PDB is α. The plane of the right-angled triangle PDp is actually perpendicular to the paper and pP sticks out towards the reader at right angles to BC. In that triangle $Pp=r\sin\alpha$, $Dp=r\cos\alpha$. The perpendicular PQ drawn from P upon the horizontal plane cannot be shown in Fig. 3 because it is in front of the paper; but its height is obviously equal to pq (Fig. 4). We note also that $\angle pDd=\angle MOS=\lambda'=\angle ODE$.

Consider the triangle PQO (Fig. 4). In that figure $\angle POQ$ is the altitude of the star, while the orientation of the base OQ gives the azimuth. These, then, are the things to be determined.

We have, from Fig. 3,

$$PQ=pq=DE+pd;$$

and the right-angled triangles DEO and BDO give

$$DE=DO\cos\lambda'=BD\tan\beta\cos\lambda'=r\tan\beta\sin\lambda.$$

* The scales of altitude should be marked, say in red ink, just inside the edges of the maps, and there should be no exterior right and left margins. The two sheets can then be laid out, at will, so as to make a single complete map.

Also $\quad pd = Dp \sin \lambda' = r \cos \alpha \cos \lambda$;

hence $\quad PQ = r \tan \beta \sin \lambda + r \cos \alpha \cos \lambda$.

Again, $\quad OP = OB = r \sec \beta$.

Hence, if the altitude of P is ϕ, we have (Fig. 4):

$$\sin \phi = PQ/OP = (\tan \beta \sin \lambda + \cos \alpha \cos \lambda)/\sec \beta,$$

i.e. $$\sin \phi = \sin \beta \sin \lambda + \cos \alpha \cos \beta \cos \lambda. \quad \text{.......(6)}$$

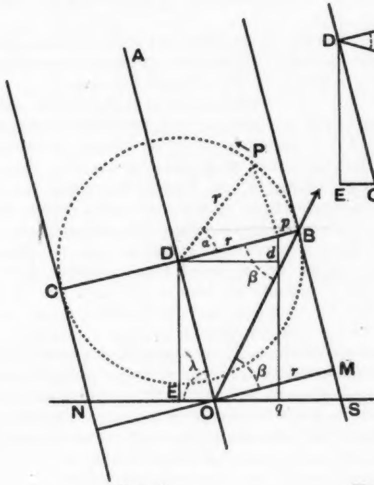


FIG. 3.

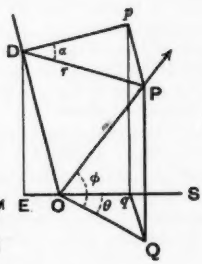


FIG. 4.

To find the azimuth we consider the right-angled triangle QOq (Fig. 4), which lies in the horizontal plane. Oq is shown in Fig. 3, but qQ stands out at right angles to the paper. Here we have (Fig. 4):

$$Qq = Pp = r \sin \alpha,$$

and in Fig. 3:

$$Oq = Eq - EO = Dd - OD \cos \lambda = Dp \cos \lambda' - r \tan \beta \cos \lambda,$$

i.e. $$Oq = r \cos \alpha \sin \lambda - r \tan \beta \cos \lambda.$$

Hence if θ be put for the azimuth $\angle SOQ$, we have:

$$\cot \theta = Oq/Qq = (\cos \alpha \sin \lambda - \tan \beta \cos \lambda)/\sin \alpha,$$

or $$\cot \theta = \cot \alpha \sin \lambda - \operatorname{cosec} \alpha \tan \beta \cos \lambda. \quad \text{.......(7)}$$

§ 15. In making the star-maps one begins by using (6) and (7) to obtain, first, a table of the azimuths θ, and then a table of the altitudes ϕ corresponding to values of α taken at intervals of 15° in combination with values of β taken at intervals of 10°. It is a fairly long business (unless divided among several trustworthy computers) but not so long as it looks; for many pairs of α and β do not appear on the map and there is much repetition of values.

The tables having been computed and used in plotting the hour-lines and declinations, it remains to insert the stars. Their positions will, as was explained in § 13, vary from map to map in accordance with the sidereal time or R.A. allocated to the meridian. *Whitaker* gives a list of the R.A. and declination of the brightest stars, but it is hardly full enough. The writer has used the list of "Mean Places of Stars" given in the official *Nautical Almanac*—omitting all whose magnitudes are below 4·5. It is a good thing to record the magnitude to the nearest integer in a small figure against each star. Also it is well to enter in succession all the stars of a particular constellation, and to mark them off with a roughly pencilled line before going on to another constellation.

The course of the ecliptic should be entered on each sheet. This is done simply, by plotting (from the tables in *Whitaker*) the sun's positions for every hour of R.A. in so far as they appear on the sheets of the maps.

A map designed to show only the stars visible at a particular sidereal time in a particular latitude may seem to have a very restricted usefulness; but a set of such maps, drawn for a series of sidereal times, is extraordinarily helpful in finding and identifying stars and constellations. The writer has made for his latitude a set of 12 maps (24 sheets) adjusted to intervals of 2 hours of sidereal time. He finds them so valuable and interesting that he is about to double the series—so as to have a map for every hour of the year. For it must be remembered that a particular sidereal hour occurs at varying solar times. Thus the map drawn for sidereal time 4 hours gives an exact picture of the heavens as they appear at about 10 p.m. on 12 December, 9.30 p.m. on 29 December, 9 p.m. on 6 January, and so on, to about 6.30 p.m. on 13 February. (This will be on the standard meridian; elsewhere the usual allowance for longitude must be made.) If this map is preceded and followed by others, a full history is presented of the rising, progress and setting of the visible stars. Stars whose altitude are above 71° 33′ cannot be shown on the sheets. These may be shown in a circle of radius 5 cm. (*i.e.* half scale) drawn on the back of one of them. It is a good thing also to write on the back a list of the dates and solar times for which the map is adapted.

V

§ 16. *Further Uses of Formulae.* Formula (6) can be used to compute the hours of rising and setting more accurately than they can be computed by (2). With the subject changed from $\sin\phi$ to $\cos\alpha$ it becomes:

$$\cos\alpha = \sin\phi \sec\beta \sec\lambda - \tan\beta \tan\lambda. \quad \text{.......................(8)}$$

If a star is actually on the horizon, the altitude $\phi = 0$, and (8) reduces to the simple formula (2) of § 9. But we saw there that a star will be visible as soon as or as long as it is not more than 34′ below the horizon. Hence the substitution $\phi = -34'$ in (8) yields the hour-angle for the star's apparent rising or setting. In the case of the sun and the moon a further allowance must be made for the breadth of the discs—which is about 32′. Visible rising and setting of the sun or the moon occurs when the upper limb of the luminary is seen on the horizon—that is, when the centre of the disc is $34' + 16' = 50'$ below the horizon.

Example. What are the corrected times of sunrise and sunset in London (lat. $51\frac{1}{2}°$ N.) on the longest day? Here we have:

$$\begin{aligned}\cos\alpha &= \sin(-50') \sec 23° \, 27' \sec 51\tfrac{1}{2}° - \tan 23° \, 27' \tan 51\tfrac{1}{2}° \\ &= \cos(180° - 55° \, 12'),\end{aligned}$$

whence α is 124° 48′, that is, 8 h. 19 m. by sidereal and 8 h. 18 m. by solar time. On 22 June the equation of time is about $-1\frac{1}{2}$ m., *i.e.* the sun is south at 12 h. $1\frac{1}{2}$ m. We conclude, then, that it rises at about 3.43 a.m. and sets at about 8.20 p.m. East or west of a standard meridian the usual allowance for longitude must be made.

The substitution of the values of α in (7) gives the azimuths of the points at which the sun will appear to rise and set. It is both instructive and interesting to make a table of these, and to predict (and verify by observation) the way in which the rising or the setting sun swings backwards and forwards along the horizon as winter gives place through spring to summer, and the year returns through autumn to winter.

On the ground that the planets are hardly ever visible on the horizon, *Whitaker* has ceased to give the times of their rising and setting, but gives, instead of these, the times at which they are 5° above the horizon. Those times may be calculated from (8), putting $\phi = +4° 50'$ to allow for the effect of refraction at altitudes of about 5°.

§ 17. *Twilight.* The *Nautical Almanac* recognises three grades of twilight, depending on the depth of the sun's centre below the horizon. First comes (*a*) civil twilight, when the depth is 6°. Outdoor work now becomes difficult and "the ploughman homeward plods his weary way". There follow (*b*) nautical twilight, when the depth is 12°, and (*c*) astronomical twilight when it is 18°. All these times may be calculated by means of (8), ϕ receiving in turn the values $-6°$, $-12°$, $-18°$. The reader should work out one or two results for one of the latitudes given in *Whitaker's* tables and so confirm his predictions.

VI

§ 18. *The Planets. Celestial Latitude.* *Whitaker* gives the R.A. and declination of the planets for dates at short intervals. If the positions are plotted on the star-maps it will be seen that they never stray far from the ecliptic. This fact hints at the importance of the ecliptic in the solar system, and suggests that it might be useful to refer the positions of the planets to it as we have already done with the sun.

With Fig. 3 before our eyes let us, then, devise one more cylinder. Its axis is to be perpendicular to the plane of the ecliptic, so that lines on the cylinder parallel to the axis will cross the ecliptic at right angles. These are of course, lines of (celestial) longitude; and angles measured up and down them from the ecliptic are called (celestial) latitude. Every such angle determines a parallel of latitude—a circle running round the cylinder in a plane parallel to its base. What we have now to inquire is how to compute a star's longitude and latitude when its R.A. and declination are given.

Formulae (6) and (7), slightly modified, supply the needed means. Let θ and ϕ now stand for longitude and latitude, β for declination, and α not for hour-angle but for R.A. Then, for the reasons we followed in deriving (4) from (1′) and (5) from (3) in §§ 11, 12, the θ of (7) must be replaced by $90° - \theta$, and the α of (6) and (7) by $90° - \alpha$. Also, if Fig. 3 were redrawn to suit the present problem, the line *OM* (equator) would be inclined to *OS* (ecliptic) at an angle $-i$, where $i = 23° 27'$. We must, therefore, replace $90° - \lambda$ in (6) and (7) by $-i$, or λ by $90° + i$. The formulae thus become:

$$\sin\phi = \sin\beta \sin(90° + i) + \cos(90° - \alpha)\cos\beta\cos(90° + i),$$

or
$$\sin\phi = \sin\beta\cos i - \sin\alpha\cos\beta\sin i, \qquad (9)$$

and
$$\cot(90° - \theta) = \cot(90° - \alpha)\sin(90° + i) - \operatorname{cosec}(90° - \alpha)\tan\beta\cos(90° + i),$$

or
$$\tan\theta = \tan\alpha\cos i + \sec\alpha\tan\beta\sin i. \qquad (10)$$

In dealing with the sun, ϕ is always zero, and (9) enables us to express $\tan\beta$ in terms of α and i. When the value is substituted in (10) that formula reduces to :

$$\tan\theta = \tan\alpha \sec i,$$

that is, to the formula (5) obtained in § 12.

To exemplify the use of (9) we employ it to find the latitude of Jupiter on a date selected at random : 1 December 1940. The R.A. was then 2 h. 21 m. (35° 15′) and the declination 12° 39′. Hence :

$$\begin{aligned}\sin\phi &= \sin 12^\circ 39' \cos 23^\circ 27' - \cos 12^\circ 39' \sin 23^\circ 27' \sin 35^\circ 15' \\ &= \sin(-1^\circ 20');\end{aligned}$$

that is, Jupiter's latitude was 1° 20′ on the negative side of the ecliptic plane. Patient repetition of this examination would show that the great planet keeps close to the ecliptic plane throughout his voyage of 11·86 years around the sun. Venus draws somewhat more freely upon the privilege of her sex, and Mercury, compared with the rest of the family, is a wild fellow.

T. P. Nunn.

GLEANINGS FAR AND NEAR.

1426. Sir,—In a game of contract bridge the other day my partner dealt and called seven hearts. The next player "passed", and so did I. The fourth player then bid seven spades. Remarking that there was no need to play out the hand, she laid down 13 spades. My partner had 13 hearts, the next player 13 diamonds, and I had 13 clubs.

My partner was Mrs. H. G. Harris, King's Acre, Braunton, North Devon, and the game took place at her house on the afternoon of Saturday, March 30. The player on her left was Miss Elsie Wright, Wayside, Thornton, and the player who made the grand slam in spades was Mrs. K. Goodban, Wayside, Thornton.

It was not a new pack of cards. They had been played with before, but for no other game than bridge. The cards were shuffled by Miss Wright in the usual way, and Mrs. Goodban cut them to Mrs. Harris, the dealer.—Yours faithfully,

(Mrs.) Marjorie S. Foster.

Chelfham, Barnstaple, N. Devon, April 9.

Our Bridge Correspondent writes :

Mathematicians tell us that the odds against a complete suit being dealt to each of four players is 2,235,197,406,895,366,368,301,560,000 to 1. One mathematician has worked it out that if 1,000,000,000 players played 100 hands a sitting for 400 sessions a year, in 50,000,000 years the chances of the occurrence in that period is only reduced to one in 100,000,000. Yet since 1927 there have been about a dozen authenticated instances in the British Empire alone. Is it that the ordinary methods of shuffling are so far removed from the mathematical haphazardness assumed in the calculations?—*Daily Telegraph*, April 13, 1940. [Per Mr. J. B. Channon.]

1427. As for Newton's speculations on the perihelion of Mercury! "Not since Shakespeare made Hector of Troy quote Aristotle", says Mr. Shaw, with great satisfaction, "has the stage perpetrated a more staggering anachronism."—*Radio Times*, February 2, 1940, on G. B. Shaw's play "In Good King Charles' Golden Days". [Per Mr. A. F. Mackenzie.]

ON SQUEEZING TABLES.

By W. Hope-Jones.

We live in days when more than ordinary care is taken to extract from jam-pots, gardens, dust-bins and ourselves the maximum of productivity; and in such times why should mathematical tables be exempt from the general squeeze? To all of us, I suppose, there come moments when we say "For this calculation I wish I had tables giving more figures": such cases occur especially when anything is found as the difference of two much larger quantities, in which a small percentage-error may appear greatly magnified in their difference (weighing the ship's cat by the displacements before and after she fell overboard). I shall outline here and attempt to justify the least laborious that I know of methods by which the inevitable approximation-errors can be cut down to a much smaller mean than is involved in using the crude values as printed.

I am not concerned here with interpolation by second or third differences to allow for the curvature of a graph: I take a short enough piece of it for curvature to be negligible, and seek only to minimise the effect of the "$\pm$ anything up to $\frac{1}{2}$" which one must imagine written after every table-entry. My instances are taken from four-figure tables of Natural Sines of small angles at 6′ intervals; but the procedure is unchanged for any kind of equal-interval table, provided that the curvature is negligible. A first test for this is that not more than two distinct differences between consecutive entries occur in the neighbourhood, as in the 5° line of a four-figure table of Sines, where the differences are 17, 17, 18, 17, 17, 18, 17, 18, 17, and the gradient evidently between $17\frac{1}{3}$ and $17\frac{1}{2}$.

A line of the table is reproduced here, with the abbreviations which will be used for the entries:

$10^4 \sin$ 5° 00′	06′	12′	18′	24′	30′	36′	42′	48′	54′
=0872	0889	0906	0924	0941	0958	0976	0993	1011	1028
a	b	c	d	e	f	g	h	i	j

I. *To adjust a single value taken from the table, without interpolation.*

Two very obvious plans are to substitute for the crude reading the mean of itself and its two or four nearest neighbours. Let us investigate the accuracy of these.

THE THREE-VALUE MEAN.

If the two differences involved are equal, clearly the three-value mean is equal to the crude reading. For instance, one's estimate of the most probable value of e is not altered by noticing that d is below it by 17 and f is above it by 17; but these observations do greatly decrease the probable error of the given 0941.

In the case of f, the fact that $g-f > f-e$ tells us at once that the true value of $f > 0958$. The three-value mean of e, f and g is $0958\frac{1}{3}$; and the relative trustworthiness of this and the crude 0958 can be assessed by comparing the root-mean-square errors (or graphically, radii of gyration) of possible values of f about the values taken.

Let $0958+y$ be the exact value of f, and let $17+x$ be the gradient.

Then the exact value of $e=0941+y-x$, and this is between $0941\pm\frac{1}{2}$.

The exact value of $g = 0975 + y + x$, and this is between $0976 \pm \frac{1}{2}$;

$$\text{thus} \quad \tfrac{1}{2} > y - x > -\tfrac{1}{2},$$
$$\text{and} \quad 1\tfrac{1}{2} > y + x > \tfrac{1}{2}.$$
$$\text{Also} \quad \tfrac{1}{2} > y > -\tfrac{1}{2}.$$

These inequalities shut in the point whose coordinates are x and y to the triangle Z in graph 1. Its centre of gravity is at $x = \frac{1}{2}$, $y = \frac{1}{6}$, which confirms the claim of $0958\frac{1}{6}$ to be the best value of f to be deduced from e, f and g.

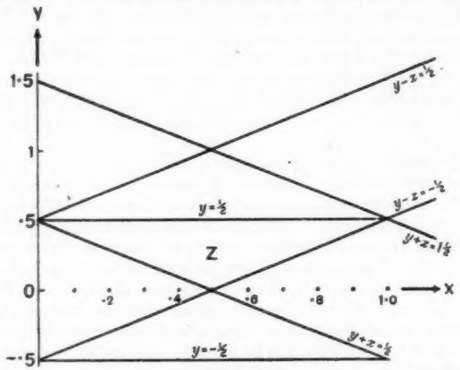


FIG. 1.

Now the (radius of gyration)^2 of a triangular lamina about a line $(y = \bar{y})$ parallel to the x-axis, through its centre of gravity, is $\frac{1}{36}\Sigma(y_1 - y_2)^2$, y_1 and y_2 and y_3 being the coordinates of its vertices. In the triangle Z this reduces to $\frac{1}{72}$.

If no reference is made to e and g, all we know about $f - 0958$ is that it is equally likely to be anything between $\frac{1}{2}$ and $-\frac{1}{2}$, so that its (radius of gyration)^2 about its mean value of 0 is $\frac{1}{12}$. So it appears that, as judged by the root-mean-error^2 test, the three-value mean is $\sqrt{6}$ times as good as the crude reading in cases where they differ.

THE FIVE-VALUE MEAN.

The complete investigation of the five-value mean (such as $\{d + e + f + g + h\}/5$ for f) is too tedious to print in full, the four differences involved, each 17 or 18, giving at first sight 16 cases to consider (somewhat decreased by symmetry and impossible combinations). The centre of gravity of the area found, as Z was above, has not always the five-value mean for its y-coordinate. The slight systematic error thus introduced, the extra labour of calculation, and the necessity of keeping the range short enough to neglect curvature safely, weigh against the utility of this; but it does give accuracy $\sqrt{10}$ times as good as the crude reading, and is sometimes worth using when the curvature is really very small.

II. *To adjust a value found by first-difference interpolation between two readings from the table.*

(It is assumed that genuine interpolation is used, not readings from a " mean difference " column, which are correct, or sometimes incorrect, only to the nearest integer.)

If we are to improve on two crude readings, such as b and c, by casting the net wider, it is natural to use two more, a and d, and to adjust b and c by reference to the four values a, b, c and d, before interpolating between them. For the benefit of those who want a ready-made result without wading through a proof, the " 1, 2, 3, 4 " rule is here stated in advance.

$$\left.\begin{array}{l}\text{" For c substitute } \cdot 1\mathrm{a}+\cdot 2\mathrm{b}+\cdot 3\mathrm{c}+\cdot 4\mathrm{d}. \\ \text{For b substitute } \cdot 4\mathrm{a}+\cdot 3\mathrm{b}+\cdot 2\mathrm{c}+\cdot 1\mathrm{d}."\end{array}\right\}$$

In the section that follows, a, b, etc., will be used to denote, not the actual numerical values 0872, 0889, etc., given above, but general values taken at equal intervals along a straight-line graph and recorded to the nearest integer. Subtracting the same integer from all, a may be taken as 0. It will further be assumed that all the differences are 0 or 1 ; and any results arrived at will be equally true if afterwards the gradient is increased by 17 or any other integer.

Then the eight possible values of a, b, c and d are :

a	b	c	d	
0	0	0	0	
0	0	0	1	(W)
0	0	1	1	(V)
0	0	1	2	(U)
0	1	1	1	(T)
0	1	1	2	(S)
0	1	2	2	(R)
0	1	2	3	

In the first and last of these eight cases it is evident that all reasonable systems of interpolation will give the same result : they are not further considered. The other six have been named W, V, U, T, S and R, and will be represented graphically, as Z was, by areas.

If the exact value of b is y, and the gradient is x, then the exact values are

$$\begin{aligned} a &= y - x, \\ b &= y, \\ c &= y + x, \\ d &= y + 2x\ ; \end{aligned}$$

and this table gives the limits between which they lie. The inequalities which form the boundaries of the areas are printed in heavier type.

	a	b	c	d
W -	$\frac{1}{2} > \mathbf{y-x} \boldsymbol{>} \boldsymbol{-\frac{1}{2}}$	$\frac{1}{2} > y > -\frac{1}{2}$	$\frac{1}{2} \boldsymbol{>} \mathbf{y+x} > -\frac{1}{2}$	$1\frac{1}{2} > \mathbf{y+2x} \boldsymbol{>} \frac{1}{2}$
V -	$\frac{1}{2} > \mathbf{y-x} \boldsymbol{>} \boldsymbol{-\frac{1}{2}}$	$\frac{1}{2} \boldsymbol{>} \mathbf{y} > -\frac{1}{2}$	$1\frac{1}{2} > \mathbf{y+x} \boldsymbol{>} \frac{1}{2}$	$1\frac{1}{2} \boldsymbol{>} \mathbf{y+2x} > \frac{1}{2}$
U -	$\frac{1}{2} > \mathbf{y-x} \boldsymbol{>} \boldsymbol{-\frac{1}{2}}$	$\frac{1}{2} \boldsymbol{>} \mathbf{y} > -\frac{1}{2}$	$1\frac{1}{2} > y+x > \frac{1}{2}$	$2\frac{1}{2} > \mathbf{y+2x} \boldsymbol{>} 1\frac{1}{2}$
T -	$\frac{1}{2} \boldsymbol{>} \mathbf{y-x} > -\frac{1}{2}$	$1\frac{1}{2} > \mathbf{y} \boldsymbol{>} \frac{1}{2}$	$1\frac{1}{2} > y+x > \frac{1}{2}$	$1\frac{1}{2} \boldsymbol{>} \mathbf{y+2x} > \frac{1}{2}$
S -	$\frac{1}{2} \boldsymbol{>} \mathbf{y-x} > -\frac{1}{2}$	$1\frac{1}{2} > \mathbf{y} \boldsymbol{>} \frac{1}{2}$	$1\frac{1}{2} \boldsymbol{>} \mathbf{y+x} > \frac{1}{2}$	$2\frac{1}{2} > \mathbf{y+2x} \boldsymbol{>} 1\frac{1}{2}$
R -	$\frac{1}{2} \boldsymbol{>} \mathbf{y-x} > -\frac{1}{2}$	$1\frac{1}{2} > y > \frac{1}{2}$	$2\frac{1}{2} > \mathbf{y+x} \boldsymbol{>} 1\frac{1}{2}$	$2\frac{1}{2} \boldsymbol{>} \mathbf{y+2x} > 1\frac{1}{2}$

In graph 2 the areas W, U, T and R are each $\frac{1}{8}$, and V and S are each $\frac{1}{4}$, of $(W+V+U+T+S+R)$. This gives the relative frequencies of the six cases under consideration.

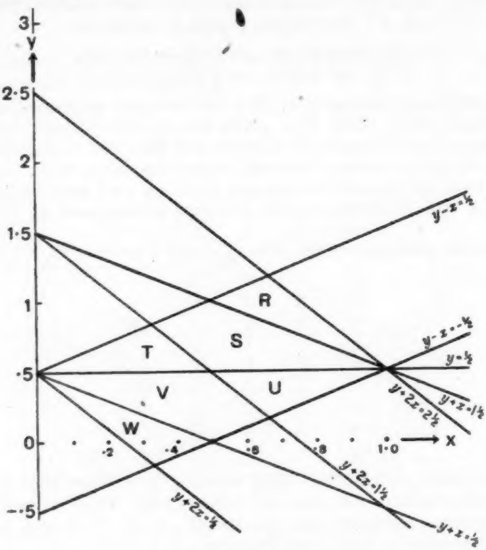


FIG. 2.

The coordinates $\bar{x}$ and $\bar{y}$ of the centres of gravity of the six areas are:

	$\bar{x}$	$\bar{y}$
W	$\frac{5}{18}$	$\frac{1}{9}$
V	$\frac{7}{18}$	$\frac{11}{36}$
U	$\frac{13}{18}$	$\frac{7}{18}$
T	$\frac{5}{18}$	$\frac{11}{18}$
S	$\frac{11}{18}$	$\frac{25}{36}$
R	$\frac{13}{18}$	$\frac{8}{9}$

It would of course be possible to devise for each area separately an adjustment-rule which would give the best value of y for that particular case; but as nobody wants the trouble of remembering six rules and investigating every time which is applicable, we would do better to aim at a rule giving the minimum mean square of error in the six cases taken together.

Let the adjusted value of b be $ka+lb+mc+nd$, $k+l+m+n$ being 1.

Then k, l, m, and n must be chosen so as to give a minimum sum of the six expressions got by substituting for a, b, c, and d in $(ka+lb+mc+nd-\bar{y})^2$, doubled in the case of V and S because their areas, representing frequencies, are twice those of W, U, T and R.

Then the expression to be made a minimum is

$$(n-\tfrac{1}{9})^2+2(m+n-\tfrac{11}{36})^2+(m+2n-\tfrac{7}{18})^2+(l+m+n-\tfrac{11}{18})^2$$
$$+2(l+m+2n-\tfrac{25}{36})^2+(l+2m+2n-\tfrac{8}{9})^2.$$

Differentiating with respect to l, m and n, we get the three equations :

$$4l+5m+7n=2\tfrac{8}{9},$$
$$5l+10m+13n=4\tfrac{7}{9},$$
$$7l+13m+20n=6\tfrac{2}{3},$$

whence $k=\frac{53}{132}$, $l=\frac{119}{396}$, $m=\frac{79}{396}$ and $n=\frac{13}{132}$.

These differ negligibly from $k=\cdot4$, $l=\cdot3$, $m=\cdot2$ and $n=\cdot1$: in fact, the expression which we have been reducing to a minimum comes to $\frac{1}{1782}$ for one set of values and to $\frac{1}{1620}$ for the other.

We therefore take the adjusted value of b as $\cdot4a+\cdot3b+\cdot2c+\cdot1d$, and of c as $\cdot1a+\cdot2b+\cdot3c+\cdot4d$. Direct interpolation between these values by accurate first differences (not the rough approximations of "mean difference" columns) now gives something very like the best possible value to be had from the four data.

III. *Degree of accuracy of interpolated value.* (1) Crude.

Let the interval between b and c be divided in the ratio $\frac{1}{2}-p:\frac{1}{2}+p$, so that first-difference interpolation gives a mixture of b and c (or their adjusted values) in the ratio $(\frac{1}{2}+p)$ of b to $(\frac{1}{2}-p)$ of c. If the crude readings of b and c are used, this comes to $(b+c)/2+p(b-c)$.

If y is the true value to which $(b+c)/2+p(b-c)$ is the approximation obtained by interpolation, and x is the gradient, then the true values of b and c are $y-(\frac{1}{2}-p)\,x$ and $y+(\frac{1}{2}+p)\,x$: so

$$b+\tfrac{1}{2} > y-(\tfrac{1}{2}-p)\,x > b-\tfrac{1}{2},$$
$$\text{and } c+\tfrac{1}{2} > y+(\tfrac{1}{2}+p)\,x > c-\tfrac{1}{2}.$$

Graph 3 shows the limits between which x and y are shut in by these inequalities.

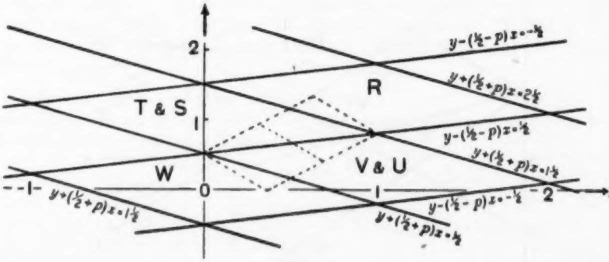


Fig. 3.

Graph 3 (on half the scale of the other graphs). The dotted lines are the new boundaries to be introduced in graph 4.

In drawing graph 3, p is taken as $\cdot 2$, but the general value p is kept in the calculations. When a and d are not being considered, cases V and U go together, and so do T and S. The large extent of the areas representing the various cases indicates the indefinite nature of our knowledge of x and y from these limited data. Each of the parallelograms W, $V+U$, $T+S$, and R has a (radius of gyration)^2 about its own $y=\bar{y}$ equal to $\frac{1}{24}+p^2/6$, which is a measure of the uncertainty of interpolation between the crude readings b and c without reference to a and d.

(2) Adjusted.

This time for b we substitute

$$\cdot 4a+\cdot 3b+\cdot 2c+\cdot 1d,$$

and for c,

$$\cdot 1a+\cdot 2b+\cdot 3c+\cdot 4d.$$

For $\frac{1}{2}(b+c)$ we now have $\frac{1}{4}(a+b+c+d)$, and for $b-c$ we have

$$\cdot 3(a-d)+\cdot 1(b-c).$$

Then interpolation gives, instead of $\frac{1}{2}(b+c)+p(b-c)$, the value

$$\tfrac{1}{4}(a+b+c+d)+p\{\cdot 3(a-d)+\cdot 1(b-c)\},$$

which will be called $\bar{\bar{y}}$.

If now y is the true value to which $\bar{\bar{y}}$ is the approximation obtained by interpolation between adjusted values, and x is the gradient, then

the true value of a is $y+(p-1\frac{1}{2})x$,
the true value of b is $y+(p-\frac{1}{2})x$,
the true value of c is $y+(p+\frac{1}{2})x$, and
the true value of d is $y+(p+1\frac{1}{2})x$.

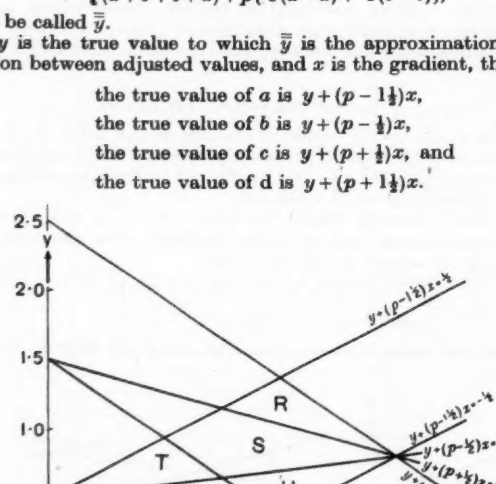


Fig. 4.

Graph 4 is an orthogonal projection of graph 2, which is the special case of it when $p=\frac{1}{2}$. It shows the areas into which x and y are shut in by the boundary-conditions W, V, U, T, S and R; and their total area is $\frac{1}{8}$ of that occupied by the corresponding areas in graph 3 in which only two data were used.

The following table gives the y-coordinates of the corners and centres of gravity of the six areas, with the squares of their radii of gyration (ρ), first about $y=\bar{y}$, and then, by adding $(\bar{y}-\bar{\bar{y}})^2$, about $y=\bar{\bar{y}}$.

					$\bar{y}$	$\bar{\bar{y}}$	ρ^2 about $\bar{y}$	ρ^2 about $\bar{\bar{y}}$
W -	$\frac{1}{2}$	$\frac{1}{4}-\frac{p}{2}$	$-\frac{p}{3}$		$\frac{1}{4}-\frac{5p}{18}$	$\frac{1}{4}-\cdot 3p$	$\frac{1}{96}+\frac{p}{72}+\frac{7p^2}{648}$	$\frac{1}{96}+\frac{p}{72}+\frac{61p^2}{5400}$
V -	$\frac{3}{4}-\frac{p}{2}$	$\frac{1}{2}$	$\frac{1}{2}-\frac{2p}{3}$	$\frac{1}{4}-\frac{p}{2}$	$\frac{1}{2}-\frac{7p}{18}$	$\frac{1}{2}-\cdot 4p$	$\frac{1}{96}+\frac{13p^2}{648}$	$\frac{1}{96}+\frac{109p^2}{5400}$
U -	$1-p$	$\frac{3}{4}-\frac{p}{2}$	$\frac{1}{2}-\frac{2p}{3}$		$\frac{3}{4}-\frac{13p}{18}$	$\frac{3}{4}-\cdot 7p$	$\frac{1}{96}-\frac{p}{72}+\frac{7p^2}{648}$	$\frac{1}{96}-\frac{p}{72}+\frac{61p^2}{5400}$
T -	$1-\frac{p}{3}$	$\frac{3}{4}-\frac{p}{2}$	$\frac{1}{2}$		$\frac{3}{4}-\frac{5p}{18}$	$\frac{3}{4}-\cdot 3p$	$\frac{1}{96}-\frac{p}{72}+\frac{7p^2}{648}$	$\frac{1}{96}-\frac{p}{72}+\frac{61p^2}{5400}$
S -	$1\frac{1}{4}-\frac{p}{2}$	$1-\frac{p}{3}$	$1-p$	$\frac{3}{4}-\frac{p}{2}$	$1-\frac{11p}{18}$	$1-\cdot 6p$	$\frac{1}{96}\quad\frac{13p^2}{648}$	$\frac{1}{96}+\frac{109p^2}{5400}$
R -	$1\frac{1}{2}-\frac{2p}{3}$	$1\frac{1}{4}-\frac{p}{2}$	$1-p$		$1\frac{1}{4}-\frac{13p}{18}$	$1\frac{1}{4}-\cdot 7p$	$\frac{1}{96}+\frac{p}{72}+\frac{7p^2}{648}$	$\frac{1}{96}+\frac{p}{72}+\frac{61p^2}{5400}$

The weighted mean of the values of ρ^2 about $\bar{\bar{y}}$ is $\frac{1}{96}+17p^2/1080$. By comparison of this with $\frac{1}{24}+p^2/6$, the value found for ρ^2 in each of the areas of graph 3, it appears that an interpolated value is from twice to 2·41 times as good if interpolated between adjusted data as if crude readings are used. The superiority of the adjusted data is less marked when used for interpolation; but the reason for this is not that adjusted data are giving less accuracy, but that the crude readings are giving more when used for interpolation than when read singly. This happens especially when the gradient is near to an integer $+\frac{1}{2}$, so that crude readings contain alternately positive and negative errors, which are decreased by interpolation. In fact, what a table does not state directly is, paradoxically, sometimes more informative than what it does.

IV. *An instance of the use of these adjustments.*

The particular question out of which this investigation arose was of the type: Find C in the Spherical Triangle in which

$$b = 33^\circ\ 31',$$

$$c = 108^\circ\ 11',$$

$$B = 35^\circ\ 20'.$$

In this case log sin C is too near to 0 to give C with reasonable accuracy by four-figure tables from the formula $\frac{\sin C}{\sin c}=\frac{\sin B}{\sin b}$, which only tells us that C

or its supplement is likely to be between 84° 15′ and 84° 27′. When this happens, C is better found from

$$4 \sin^2 (45° - \tfrac{1}{2}C) \sin b = 2 \sin b + \cos (c+B) - \cos (c-B).*$$

In this case
$$\left.\begin{aligned} c+B &= 143° \ 31' \\ c-B &= \ \ 72° \ 51' \end{aligned}\right\}$$

and $\cos (c-B) - \cos (c+B)$ is near enough to $2 \sin b$ to bring us up against the "ship's cat" difficulty. The tables must therefore be squeezed for more than ordinary accuracy until a decent approximation to

$$2 \sin b + \cos (c+B) - \cos (c-B)$$

has been obtained.

		× 4, 3, 2, 1	× 1, 2, 3, 4
sin 33° 24′ =	·5505	20	5
30′	19	57	38
36′	34	68	102
42′	48	48	192
		193	337

Hence adjusted by the "1, 2, 3, 4 rule",

$$\left.\begin{aligned} \sin 33° \ 30' &= \cdot 55193 \\ \sin 33° \ 36' &= \cdot 55337 \end{aligned}\right\} \text{ Difference} = 144.$$

Thus sin 33° 31′, by interpolation between adjusted values,

$$\begin{aligned} &= \cdot 55193 + 24 \\ &= \cdot 55217, \end{aligned}$$

and so
$$2 \sin b = 1 \cdot 10434.$$

Similarly adjusted and adjusted
$$\left.\begin{aligned} \sin 53° \ 30' &= \cdot 80386 \\ \sin 53° \ 36' &= \cdot 80489 \end{aligned}\right\} \text{ Difference} = 103.$$

and
$$\begin{aligned} \cos (c+B) &= -\sin 53° \ 31' = -\ \ \cdot 80403\tfrac{1}{6} \\ -\cos (c-B) &= -\cos 72° \ 51' = -\ \ \cdot 29487\tfrac{1}{2} \\ & \qquad\qquad\qquad\quad -1 \cdot 09890\tfrac{2}{3} \\ 2 \sin b &= \qquad\qquad\quad\ \ 1 \cdot 10434 \\ 4 \sin^2 (45° - \tfrac{1}{2}C) \sin b &= \qquad\qquad\quad\ \ \cdot 00543\tfrac{1}{3} \end{aligned}$$

At this point the need for squeezing the tables disappears, and we continue the calculation using them in the ordinary way.

$$\sin^2 (45° - \tfrac{1}{2}C) \sin b = \frac{\cdot 00543\tfrac{1}{3}}{4} = \cdot 00135\tfrac{5}{6}.$$

N	L
$\cdot 00135\tfrac{5}{6}$	$\bar{3} \cdot 1330$
$\div \sin b$	$\bar{1} \cdot 7421$
$\sin^2 (45° - \tfrac{1}{2}C)$	$\bar{3} \cdot 3909$
$\pm \sin (45° - \tfrac{1}{2}C)$	$\bar{2} \cdot 6954\tfrac{1}{2}$

$$45° - \tfrac{1}{2}C = \pm 2° \ 50\tfrac{1}{2}'. \quad C = 90° \mp 5° \ 41' = 84° \ 19' \text{ or } 95° \ 41'.$$

* Not given in any books which I have seen; but this is slender evidence for claiming it as a new discovery.

The size of the errors involved in this work is shown by comparison with calculation by seven-figure tables.

$$
\begin{array}{rl}
\cos(c+B) = & -\ \cdot 8040299 \\
-\cos(c-B) = & -\ \cdot 2948743 \\
\hline
 & -\ 1\cdot 0989042 \\
2\sin b = & -\ 1\cdot 1043590 \\
\hline
\end{array}
$$

$$
\begin{array}{r|l}
N & L \\
\cdot 0054548 \div 4 = \cdot 0013637 & \bar{3}\cdot 1347188 \\
\div \sin b & \bar{1}\cdot 7420803 \\
\hline
\sin^2(45^\circ - \tfrac{1}{2}C) & \bar{3}\cdot 3926385 \\
\hline
\pm\sin(45^\circ - \tfrac{1}{2}C) & \bar{2}\cdot 69631 92\tfrac{1}{2} \\
\hline
\end{array}
$$

$45^\circ - \frac{1}{2}C = \pm 2^\circ\ 50'\ 54\frac{1}{2}''$. $C = 90^\circ \mp 5^\circ\ 41'\ 49'' = 84^\circ\ 18'\ 11''$ or $95^\circ\ 41'\ 49''$.

The angle C has therefore been found from the four-figure tables with an error of $49''$.

Opinions will differ as to what constitutes " ordinary care " in the normal use of tables ; but if crude readings are used and interpolation is taken to the nearest integer in the 4th place, using halves when two integers are equally likely, the $\sin^2(45^\circ - \frac{1}{2}C)$ method gives in this case $84^\circ\ 21'$ as the nearest minute, and this is in error by $2'\ 49''$. W. HOPE-JONES.

1428. The following extracts are from *Spanish Testament* by Arthur Koestler. In all of these extracts the author refers to his imprisonment by the rebels.

The first day in prison began ; the first of one hundred and two days. . . . I took a piece of wire out of the bedstead and began to scrawl mathematical formulae on the wall. I worked out the equation of an ellipse ; but I couldn't manage the equation of a hyperbola. The formulae became so long that they reached from the W.C. to the wash-basin.—Second Part ; *Dialogue with Death*, p. 240.

[After his transfer from Malaga prison to that of Seville.] Later on I busied myself with the equation of a hyperbola again . . . the rest of the afternoon was spent in mathematics, reciting poetry and the Trojan war.—*Ibid.*, pp. 276, 277.

Every man needs a different pill to help him to arrive at a *modus vivendi* with his misery. . . . I, too, had my pills, a whole collection of various sorts of them, from the equation of a hyperbola . . . to every kind of synthetic product of the spiritual pharmacy.—*Ibid.*, pp. 288, 289.

Time crawled through this desert of uneventfulness as though paralysed in both feet. . . . It seems that those days which, owing to their uneventfulness and dreariness, seem longest, shrink to nothing as soon as they have become the past, precisely because of their uneventfulness. In the perspective of the past they have no extension, no volume, no specific gravity ; they become geometric points, a diminishing vacuum, nothing.—*Ibid.*, p. 291.

I fancy there must be some exact mathematic relationship ; one's disbelief in death grows in proportion to its approach.—*Ibid.* p. 310.

The usual idea of prison life can be expressed in the form of an equation :

Prison life is equal to normal life minus freedom. This equation is all wrong. Stated correctly, it should be : normal life bears the same relation to prison life as life on the earth to life on the moon. Incomparable magnitudes are involved. . . .—*Ibid.*, p. 373. [Per Mr. P. J. Harris.]

1429. But the whole ballet was dominated by Nijinsky's performance as the Gold Negro. I can still see his astounding first entrance when, . . . , he darted towards Zobeida with the swiftness of a bolt from a crossbow, a flashing parabola, to stretch his arms wide in a gesture of possession. . . .—C. W. Beaumont, *Complete Book of Ballets*, p. 709. [Per Mr. P. J. Harris.]

EXPLAINING THE CONSTRUCTION OF LOGARITHM TABLES TO 3RD FORMERS AND A NOTE ON *e*.*

BY S. INMAN.

WHETHER it is because of the utility aspect or whether it is because of some other reason, boys are invariably interested in logarithms. Whenever I take up the subject with a form, at least one boy will ask: " Please, Sir, how were logarithm tables constructed?" When a master is confronted with this question, he usually thinks of logarithmic expansions, and he is obliged to tell the boy that the explanation is beyond his understanding.

The first attempt at a simple explanation that I came across was in *Mathematics for the Million* by Hogben. Hogben points out that $2^{10} = 1024 \simeq 10^3$. Thus $10 \log 2 \simeq 3$ and $\log 2 \simeq 0{\cdot}300$. Again $81 \simeq 80$. Thus

$$4 \log 3 \simeq 1 + 3 \log 2 = 1{\cdot}9 \quad \text{and} \quad \log 3 \simeq 0{\cdot}475.$$

Hogben does not develop the subject much farther. Nevertheless, his explanation is simple and depends on elementary principles. The values he obtains, however, are correct to only 2 places of decimals.

A few months ago I tried to obtain an explanation of the construction of logarithm tables, using only elementary principles, and this paper is the outcome of my efforts. I developed several methods, but there was an idea common to them all. This made use of a factor which might be called a *converging factor*. A number which was close to a power of 10 was divided by that power of 10 to bring it near to 1. At first I could not get sufficiently close to 1, but by combining two numbers each close to 1, I got a much better approximation to 1. Thus if $x = 0{\cdot}999 = 3^3 . 37/10^3$, and $y = 1{\cdot}001 = 7 . 11 . 13/10^3$. Each is close to 1, but their product, 0·999999, is much closer. Thus

$$3^3 . 7 . 11 . 13 . 37/10^6 \simeq 1.$$

Taking logarithms of both sides, we get

$$3 \log 3 + \log 7 + \log 11 + \log 13 + \log 37 \simeq 6.$$

Hence we can calculate log 37 if we know the logs of 3, 7, 11, 13. Here x or y is a converging factor when applied to the other. Here is another example: $x = 399/400 = 3 . 7 . 19/2^2 . 10^2$ and $y = 1599/1600 = 41 . 3 . 13/2^4 . 10^2$. If $x = 1 - a$ and $y = 1 - b$, $a = 4b$. Thus we take x/y^4 so that $x/y^4 = 0{\cdot}999998$. Hence $\log x - 4 \log y = 0$. This gives us log 41 if we know the logs of 2, 3, 7, 13, and 19. It is seen that if $x = 1 + a$ and $y = 1 + b$, we divide if a and b have the same sign and multiply if they have opposite signs.

If the above is to be used as a method, two questions arise. (1) Can we get converging factors as above to give the logarithm of any prime in terms of the logarithms of smaller primes? (2) Can we calculate the logarithms of 2 and 3 so that the above method can be used? With regard to (1), I shall show that there is no difficulty in obtaining converging factors for any prime. As regards (2), log 2 and log 3 can be calculated as follows:

log 2 $\quad x = 1{\cdot}024 = 2^{10}/10^3$; $x^{10} = 1{\cdot}2676506 \ldots$;

$y = x^{10} \times 0{\cdot}8 = 2^{103}/10^{31} = 1{\cdot}0141205 \ldots$; $y^{16} = 1{\cdot}2512053 \ldots$;

$z = y^{16} \times 0{\cdot}8 = 2^{1651}/10^{497} = 1{\cdot}001204 \ldots \simeq 1.$

Hence $1651 \log 2 \simeq 497$ and $\log 2 = \underline{0{\cdot}3010297}$ (correct value $= 0{\cdot}3010300$).

* A paper read to the London Branch of the Mathematical Association.

log 3 $a=80/81$; $x=0{\cdot}243\times 4=0{\cdot}972=3^5\,.\,2^2/10^3$;
$y=a^2x=159423/1600000=3^{13}/2^4\,.\,10^5$;
$x/y^8=1{\cdot}000033\simeq 1$. By taking logs of both sides and simplifying, we obtain $99\log 3-34\log 2=37$.

Hence $\log 3=\underline{0{\cdot}4771214}$ (correct value $=0{\cdot}4771213$).

Primes less than 100. We are now in a position to calculate the logs of all the primes in succession. It is sufficient if this is done for primes less than 100. In each case two numbers, x and y, each nearly 1, are combined to form a number much nearer to 1. The results of the various calculations are summarised in Table A overleaf. The first column gives the prime. The factors of x and y in the next two columns consist of the prime and lower primes. The next column shows how x and y are combined to give a close convergent to 1. The log of the prime which results from this is in the next column and the true value of the logarithm is in the last column. The logarithms in the last two columns agree in all cases except for 53, which is 1 out in the 5th decimal place.

It will be seen that after the first half-dozen primes or so the convergents in column 4 become very close to 1. Had the earlier logs been calculated more accurately, the remaining logarithms could easily have been calculated to 6 or even 7 places of decimals. The numbers in the x columns are easy to obtain. Thus, if n is prime, the highest prime factor of $(n+1)$ and $(n-1)$ is in each case less than n. Hence $n^2/(n^2-1)$ is of the form $1+a$ where a is small, and does not involve a prime greater than n. This idea is made use of in calculating the logarithms of 11, 17, 19 and several other primes. The calculation of log 79 in Table A illustrates how small a can be by this method when the prime is large. If $n=101$, $n^2/(n^2-1)=1{\cdot}00001$, which is already very near to 1 without the aid of the factor y. After a while, x becomes self-sufficient and we can dispense with the factor y. Thus,

$$1000^2/(1000^2-1)=1{\cdot}000001$$

is a sufficiently good convergent to 1, involves the primes 3, 7, 11, 13, 37, and enables us to obtain log 37. $1001^2/(1001^2-1)=1{\cdot}000001$ and involves the primes 3, 7, 11, 13, 167. It leads to the log of 167.

$$10000^2/(10000^2-1)=1{\cdot}00000001$$

is exceedingly close to 1. It involves the primes of 3, 11, 101, 73, 137, and leads to the value of log 137. In the expression $n^2/(n^2-1)$, n need not be near to a power of 10; it is merely necessary for n to be large to give a good convergent to 1. I feel I am only a stranger in Number Town and that anyone more familiar with its streets could produce some interesting results by the method just described.

It is usual at this point to complete the explanation of the calculation of logarithm tables by explaining how to obtain the logarithms of composite numbers. Any explanation which did not go beyond this, however, would be seriously incomplete. It makes no mention of the most important part in the calculation of mathematical tables. I refer to the method of interpolation, of which I shall give a simple explanation. Without interpolation the calculation of tables would be so formidable as to be almost impracticable.

Three-figure numbers. The first step in the calculation of the logarithms of the 3-figure numbers is to calculate the logarithms of the composite numbers less than 100. Having, now, the logarithms of the numbers from 1 to 100, we take their successive differences. From these we shall calculate the logarithms of 100 to 999. It would seem easy enough to calculate the log-

TABLE A.

Prime	x	y	Combination of x and y	Log of Prime obtained	Actual Value
5			$1-\log 2$	0·69897	0·69897
7	$\frac{2401}{2400}=\frac{7^4}{2^3.3.10^2}$	$1{\cdot}00842=\frac{7^5.3.2}{10^5}$	$\frac{y}{x^{20}}=1{\cdot}00005$	0·84510	0·84510
11	$\frac{121}{120}=\frac{11^2}{2^2.3.10}$	$\frac{81}{80}=\frac{3^4}{2^3.10}$	$\frac{x^3}{y^2}=1{\cdot}00005$	1·04139	1·04139
13	$\frac{1001}{1000}=\frac{7.11.13}{10^3}$	$\frac{126}{125}=\frac{7.3^2.2^4}{10^3}$	$\frac{x^8}{y}=1{\cdot}00003$	1·11394	1·11394
17	$\frac{289}{288}=\frac{17^2}{2^5.3^2}$	$\frac{144}{143}=\frac{2^4.3^2}{11.13}$	$\frac{y}{x^2}=1{\cdot}00004$	1·23045	1·23045
19	$\frac{361}{360}=\frac{19^2}{2^2.3^2.10}$	$\frac{121}{120}=\frac{11^2}{2^2.3.10}$	$\frac{x^2}{y}=1{\cdot}00002$	1·27875	1·27875
23	$\frac{529}{528}=\frac{23^2}{2^4.3.11}$	$\frac{133}{132}=\frac{7.19}{2^2.3.11}$	$\frac{x^4}{y}=1{\cdot}00002$	1·36173	1·36173
29	$\frac{841}{840}=\frac{29^2}{2^2.3.7.10}$	$\frac{209}{210}=\frac{11.19}{3.7.10}$	$x^4y=0{\cdot}99999$	1·46240	1·46240
31	$\frac{961}{960}=\frac{31^2}{2^5.3.10}$	$\frac{121}{120}=\frac{11^2}{2^2.3.10}$	$\frac{x^6}{y}=1{\cdot}00002$	1·49136	1·49136
37	$\frac{1369}{1368}=\frac{37^2}{2^3.3^2.19}$	$\frac{341}{342}=\frac{11.31}{2.3^2.19}$	$x^4y=0{\cdot}999995$	1·56820	1·56820
41	$\frac{1599}{1600}=\frac{41.13.3}{2^4.10}$	$\frac{399}{400}=\frac{19.3.7}{2^2.10}$	$\frac{x^4}{y}=1{\cdot}000002$	1·61278	1·61278
43	$\frac{1849}{1848}=\frac{43^2}{2^3.3.7.11}$	$\frac{925}{924}=\frac{37.10^2}{2^3.3.7.11}$	$\frac{x^2}{y}=1{\cdot}000000$	1·63347	1·63347
47	$\frac{799}{800}=\frac{47.17}{2^3.10^2}$	$\frac{399}{400}=\frac{3.7.19}{2^2.10^2}$	$\frac{x^2}{y}=1{\cdot}000002$	1·67210	1·67210
53	$\frac{2809}{2808}=\frac{53^2}{2^3.13.3^3}$	$\frac{702}{700}=\frac{2^2.3^3.13}{7.10^2}$	$\frac{y}{x^8}=1{\cdot}000005$	1·72427	1·72428
59	$\frac{3481}{3480}=\frac{59^2}{29.2^2.3.10}$	$\frac{1749}{1750}=\frac{53.3.11.2^2}{7.10^3}$	$x^2y=1{\cdot}000000$	1·77085	1·77085
61	$\frac{3599}{3600}=\frac{59.61}{2^2.3^2.10^2}$	$\frac{399}{400}=\frac{3.7.19}{2^2.10^2}$	$\frac{x^9}{y}=1{\cdot}000002$	1·78533	1·78533
67	$\frac{4489}{4488}=\frac{67^2}{11.17.2^3.3}$	$\frac{225}{224}=\frac{3^2.10^2}{2^7.7}$	$\frac{x^{20}}{y}=1{\cdot}000003$	1·86207	1·86207
71	$\frac{5041}{5040}=\frac{71^2}{2^3.3^2.7.10}$	$\frac{1001}{1000}=\frac{7.11.13}{10^3}$	$\frac{y}{x^5}=1{\cdot}000008$	1·85126	1·85126
73	$\frac{803}{800}=\frac{73.11}{2^3.10^2}$	$\frac{799}{800}=\frac{47.17}{2^3.10^2}$	$y^2x=0{\cdot}99999$	1·86332	1·86332
79	$\frac{6399}{6400}=\frac{79.3^4}{2^6.10^2}$	$\frac{799}{800}=\frac{47.17}{2^3.10^2}$	$\frac{y}{x^8}=0{\cdot}999999$	1·89763	1·89763
83	$\frac{249}{250}=\frac{83.2^2.3}{10^3}$	$\frac{125}{124}=\frac{10^3}{2^5.31}$	$x^2y=1{\cdot}000016$	1·91908	1·91908
89	$\frac{801}{800}=\frac{89.3^2}{2^3.10^2}$	$\frac{201}{200}=\frac{3.67}{2.10^2}$	$\frac{x^4}{y}=1{\cdot}000009$	1·94939	1·94939
93	$\frac{1302}{1300}=\frac{2.7.93}{13.10^2}$	$\frac{1296}{1300}=\frac{3^5.2^2}{13.10^2}$	$x^2y=0{\cdot}99999$	1·96848	1·96848
97	$\frac{9216}{9215}=\frac{2^{11}.3^2}{10.19.97}$	$\frac{2299}{2300}=\frac{11^2.19}{23.10^2}$	$x^4y=1{\cdot}000000$	1·98677	1·98677

arithms in the line commencing, say 96. I shall therefore take a bad part of the table, namely the line commencing 31. Table B below gives part of the log differences mentioned above.

TABLE B.

No.	Log	Difference	Difference per unit in 3rd figure	Decreases	
30	47712				
		1424	142·4		
31	49136			4·5	Average = 4·4 after 10 logs ;
		1379	137·9		∴ successive differences
32	50515			4·3	decrease by 0·44.
		1336	133·6		
33	51851				

Since the average difference in the line 310 is 137·9, this can be taken to be the difference between the logarithms at the extremities of a unit interval centred at 315. This is the interval x in Fig. 1. In this figure the log dif-

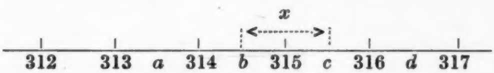


FIG. 1.

ferences for the intervals *a*, *b*, *c*, etc., decrease by 0·44. The interval b is only half a unit to the left of interval x. Hence the log difference for interval b is $0{\cdot}22 + 137{\cdot}9$, *i.e.* 138·(12). Adding (and subtracting) 0·44 in succession we get the log differences for the remaining intervals to the left (and right) of b. These are in row A of Table C. The numbers at the head of the columns are those at the beginning of the intervals. As log 3·10 is 0·49136, we add the differences in succession and thus obtain the logarithms of the numbers 311 to 319. These are given in row B and are rounded off in row C. Row D gives the correct values, which are in full agreement with those in row C.

TABLE C.

	310	311	312	313	314
A	139·88	139·44	139·00	138·56	138·12
B	49136	49275(88)	49415(32)	49554(32)	49692(88)
C	49136	49276	49415	49554	49693
D	49136	49276	49415	49554	49693

	315	316	317	318	319
A	137·68	137·24	136·80	136·36	—
B	49831(00)	49968(68)	50105(92)	50242(72)	50379(08)
C	49831	49969	50106	50243	50379
D	49831	49969	50106	50243	50379

4th figure differences. The 4th figure differences are easily obtained. Since the average difference for unit interval in the 3rd figure is 137·9, that for unit interval in the 4th figure is 13·79. Multiplying this by 1, 2, 3, ... 9, we obtain the mean 4th figure differences, which are rounded off in Table D.

TABLE D.

4th figure	1	2	3	4	5	6	7	8	9
Mean difference	14	28	41	55	69	83	97	110	124

These figures are in agreement with those of printed tables. The methods summarised in Tables C and D were also tested in the lines commencing 91, 51 and 21. In all cases the calculated logarithms were found to have as good a degree of reliability as those in line 31.

Lines commencing 10 *to* 20. The logarithms of all the even numbers between 100 and 200 can be easily calculated by adding log 2 to the logs of the numbers between 50 and 100. As log 2 has the simple value 0·30103 this takes very little time. The logs of the remaining three-figure numbers can then be obtained by interpolation. This was tried in the very worst row of the table, the line commencing 100. The details of the interpolation are omitted, but the results are summarised in Table E.

TABLE E.

Number	101		103		105		107		109
Calculated log	00433		01264		02120		02939		03743
Correct value	00432		01264		02119		02938		03743

4th figure	1	2	3	4	5	6	7	8	9
Calculated value (100 to 104)	42	85	127	170	212	254	297	339	381
Correct value	42	85	127	170	212	254	297	339	381

The 4th figure differences are all correct and three of the other numbers are one out in the 5th figure. This however is due to working to five figures and is not due to a defect in the method.

The calculation of the whole table of logarithms is thus accounted for. The interesting thing is that the number of logarithms calculated from first principles is only 24. From these 24 the whole table of logarithms can be deduced to give the logarithms of numbers between 1 and 100,000. This works out as 4000 derived logarithms for every one calculated from first principles. The fact that so few have to be calculated from first principles will excuse any lack of method in their calculation.

The method just described brings out in a striking way the importance of the method of interpolation. This is the best way of calculating not only logarithm tables but also mathematical tables in general. In fact, without its use, the calculation of mathematical tables is impracticable. It seems likely that interpolation will have its place in some future mathematical curriculum. Briggs, himself, developed formulae for interpolating between the logarithms he obtained from first principles. He was the first to use the method of finite differences and his formulae are equivalent to those in use at the present day. The interpolations which I developed in an informal way give the same results as would have been given by Briggs's formulae.

It may now be considered that I have provided an answer to the boy who asked how logarithms were constructed. The worst of providing an answer to an inquisitive boy is that the answer only forms the excuse for another question. I have no doubt that the questioner will say: "I understand how logarithms can be calculated to 5 places of decimals, but can they be calculated to any degree of accuracy? If so, how?" The following is an answer to this question; it also, to a large extent, replaces much of the early part of this paper. I took the square root of 10, then the square root of the answer, and the square root of this answer, and so on. I thus obtained Table F as follows:

TABLE F.

TABLE OF CONVERGING FACTORS (OR RADICES).

n	$x_n = 10^{1/2^n}$	$\log x_n$	n	$x_n = 10^{1/2^n}$	$\log x_n$
1	3·16227,76602	0·5	18	1·00000,87837	0·00000,38147
2	1·77827,94100	0·25	19	1·00000,43918	0·00000,19073
3	1·33352,14322	0·125	20	1·00000,21959	0·00000,09537
4	1·15478,19847	0·0625	21	1·00000,10980	0·00000,04768
5	1·07460,78283	0·03125	22	1·00000,05490	0·00000,02384
6	1·03663,29284	0·01562,5	23	1·00000,02745	0·00000,01192
7	1·01815,17217	0·00781,25	24	1·00000,01372	0·00000,00596
8	1·00903,50448	0·00390,625	25	1·00000,00686	0·00000,00298
9	1·00450,73643	0·00195,3125	26	1·00000,00343	0·00000,00149
10	1·00225,11483	0·00097,65625	27	1·00000,00172	0·00000,00075
11	1·00112,49414	0·00048,82813	28	1·00000,00086	0·00000,00037
12	1·00056,23126	0·00024,41406	29	1·00000,00043	0·00000,00019
13	1·00028,11168	0·00012,20703	30	1·00000,00021	0·00000,00009
14	1·00014,05485	0·00006,10352	31	1·00000,00011	0·00000,00005
15	1·00007,02718	0·00003,05176	32	1·00000,00005	0·00000,00002
16	1·00003,51353	0·00001,52588	33	1·00000,00003	0·00000,00001
17	1·00001,75675	0·00000,76294	34	1·00000,00001	0·00000,00001

The calculation of the table does not take as long as it would seem. After $n = 17$, it is merely necessary to halve the decimal part of the square root to obtain the next square root, and before that the working becomes progressively easier. The table forms a set of converging factors and their logarithms, with the factors getting closer and closer to 1. The table is in fact a machine for turning out logs. A number is put in at one end of the machine, and as it passes through the machine, the grinding becomes finer and finer until the logarithm of the number emerges at the other end. Table G illustrates how the process is used to calculate $\log_{10} 2$. We start with $1{\cdot}024 = 2^{10}/10^3$. This is divided by the nearest x below, *i.e.* x_7. The quotient which is nearer to 1 than 1·024 is divided by x_8 and the next quotient is still nearer to 1. The process is continued until the final quotient is 1·00000,00000. Thus $1{\cdot}024 = abc \ldots k$ and

$$10 \log 2 - 3 = \log a + \log b + \log c \ldots + \log k = 0{\cdot}01029999567.$$

So $\log 2 = 0{\cdot}30102{,}99957$, which is correct to 10 places of decimals.

After the division by x_{18} it is merely necessary to subtract the decimal part of the divisor from the dividend to get the next quotient. In fact, 2 or 3 divisions can be done simultaneously so that the last 5 divisions do not take more than two minutes. Also, the first five divisions become progressively easier. Table F can deal with any number in however crude a form. Thus to find the log of 631, divide by 10^2 to obtain 6·31. This number then fits

into the machine. It is divided by x_1 to give 1·995 ... ; this is divided by x_2 to give 1·122 ... , and so on. It is obviously better to start with a number of the form $1+a$ where a is small, and numbers of the form of the x-values of Table A would save much time. The simpler combinations of x and y in that table would save even more time. It is possible to converge to 1 upwards from a value less than 1. In this case the converging factors are used as multipliers.

TABLE G.

DIVISOR	QUOTIENT	LOG OF DIVISOR
$a=x_7\ \ =1{\cdot}01815,17217$	1·00574,40145	0·00781,25
$b=x_9\ \ =1{\cdot}00450,76343$	1·00123,11012	0·00195,3125
$c=x_{11}=1{\cdot}00112,49414$	1·00010,60405	0·00048,82819
$d=x_{15}=1{\cdot}00007,02718$	1·00003,57662	0·00003,05176
$e=x_{16}=1{\cdot}00003,51353$	1·00000,06309	0·00001,52583
$f=x_{22}=1{\cdot}00000,05490$	1·00000,00819	0·00000,02384
$g=x_{25}=1{\cdot}00000,00686$	1·00000,00133	0·00000,00298
$h=x_{28}=1{\cdot}00000,00086$	1·00000,00047	0·00000,00037
$j=x_{29}=1{\cdot}00000,00043$	1·00000,00004	0·00000,00019
$k=x_{32}=1{\cdot}00000,00005$	1·00000,00000	0·00000,00002
	10 log 2 − 3	= 0·01029,99567
	Hence log 2	= 0·30102,99957

As far as a class explanation is concerned, Table F to 6 decimal places is all that is necessary. This would occupy less than a dozen lines, and yet it would contain a complete explanation of how to calculate the logarithm of any number. The calculation of log 2 from 1·024 by this short table takes about 10 minutes and gives 0·3010300.

It is necessary to add some further detail, largely of an historical character. To omit this would lead to a false impression. I came up against some historical facts in the first instance when seeking a recorded value of log 2 to check my calculation, and I gradually delved more deeply into the subject. For some of these facts I am indebted to my colleague Mr. Bennett, who some years ago wrote a thesis on the History of Logarithms. Although I evolved, independently, everything which I have described in this paper, I came to realise that much of the way along which I passed had been trodden by other people. Thus I learned that Weddle, in 1845, used converging factors of the form $1-(\cdot 1)^r n$, $n=1, 2, \ldots 9$, and, in 1847, Hearn used factors of the form $1+(\cdot 1)^r n$, $n=1, 2, \ldots 9$. These were known as Weddle and Hearn numbers respectively. They have two advantages over the factors of Table F. Being more closely packed, they give a more rapidly convergent set of divisors (or in the case of Weddle, multipliers). Also, the simple form of the divisors enable the divisions to be done more rapidly. On the other hand, the calculation of the logarithms of these factors is a problem yet to be solved, whereas the calculation of the logarithms of the factors of Table F is an obvious process. Also the method I have described is the simplest complete explanation.

Weddle and Hearn provide examples of discoveries being all but forgotten and being rediscovered all over again. Weddle and Hearn numbers were far from being new. They were discovered and rediscovered several times previously. Attwood, of Attwood's machine, rediscovered them in 1786. If we go back far enough, we find the method described by Briggs in 1624 in his book *Arithmetica Logarithmica*, in which he gives the logarithms of $1+(\cdot 1)^r n$, $r=1, 2, \ldots, 9$, $n=1, 2, \ldots, 9$, to 14 places of decimals. Briggs by a kind of

contracted division combined the whole succession of divisions into one process. The method was rediscovered with improvements by Robert Flower in 1771. He called his method the Radix method, and the name radices, which he gave to his converging factors, is the name by which such factors are generally known. Flower's tract, which contained the logarithms of his factors to 21 decimal places, found its way to the Continent and from the beginning of the nineteenth century onwards inspired a succession of Radix calculators. The best table of radices was compiled by Gray in 1876; they were of the form $1+(\cdot 001)^r n$, $n=1, 2, \ldots, 999$. Gray calculated the logarithms of these to 24 decimal places, and with these, logarithms can be calculated with great rapidity.

Recently I examined the index of *The Mathematical Gazette* 1894–1931, and the large number of articles and notes on logarithms reflected the unceasing interest in the subject since the time of Napier. Among them was a half-page note by P. L. Hall. This note (Oct. 1924) gives Table F to 6 decimal places (most of these were 1 to 3 out in the 6th place). Mr. Hall used the method with success in several forms, and this is interesting in view of the title of this paper. Mr. Hall stated that he based his method on the work of Professor Perry and others. Further investigation showed that the method was discovered still earlier. In 1783 Callet had calculated the values of $10^{1/2^n}$ up to $n=120$ and had published the first 60 and their logarithms. The method may have been known still earlier, possibly by Briggs. Briggs certainly calculated the continued square root of 10 up to the 54th but used these as the basis of a totally different and much longer method.

As I have confined myself to the base 10, I have made little mention of Napier and would therefore appear to underestimate his work. Napier's work was of course of the highest value from every point of view, particularly the functional aspect. Napier had given careful thought to the base 10, but his health had commenced to fail. He discussed his methods freely and frankly with Briggs, indeed with the hope that Briggs would make full use of them, and undoubtedly Briggs owed much to Napier.

There are two impressions which have stood out clearly. One is that there are many people who are keenly desirous of obtaining the information outlined in this paper but are unaware of its existence. The other is that, owing to the negligible dispersion of this knowledge, much unnecessary research has been done over and over again to win knowledge already discovered which should have been common property. A wider publicity would therefore achieve a double remedy, and it is hoped that this paper will help to bring this about.

For those who are interested in the Project Method, the subject of this paper seems worthy of study. In any case it offers a rich harvest in the way of mathematical ideas. It deals with an infinite product which converges. The important subject of interpolation is introduced. Approximations, which are associated with so many different parts of mathematics, are touched upon frequently. Another topic, ratio, receives full prominence, and is also closely associated with the subject of logarithms. Use is made of the approximation formulae $1/(1+x) \simeq 1-x$, $(1+a)/(1+x) \simeq 1+a-x$, $\sqrt{(1+x)} \simeq 1+\frac{1}{2}x$, where x and a are small. The binomial $(1+x)$, where x is small, which was used freely, is essential in binomial and, appropriately, logarithmic expansions.

The base e. In using Table F, logarithms are "ground" out in unequal steps, many of them coarse. Let them be "ground out" in equal but very fine steps. A single "fine" factor will do for "grinding out" the logarithms of all numbers. This factor is equivalent to the form $(1+1/n)$ where n is very large. The number of such factors needed to give the logarithm of a

number N is very large, say kn. k must then be a measure of log N. Let us call it the log of N. We have lost the base 10 and have a new base which we can call " e ". We have $N=(1+1/n)^{kn}$ and $\log N=k$. We can now find $\log_e 10$. From Table F we have $x_{18}=1{\cdot}0000087837$. Thus

$$10=(1+0{\cdot}0000087837)^{2^{18}} \quad \text{and} \quad 2^{18}=kn \quad \text{where} \quad 1/n=0{\cdot}0000087837$$

and so $$k=\log_e 10=2^{18}\times 0{\cdot}0000087837=2{\cdot}30260$$

(correct value is 2·30259). To obtain a better value of $\log_e 10$ we must take a larger value of n. We have much larger values in Table F, but these, unfortunately, are to few significant figures. Briggs gives

$$x_{54}=1{\cdot}00000,00000,00000,12781,91493,20082\ldots.$$

Hence $$\log_e 10=(x_{54}-1)2^{54}=2{\cdot}30258,50929,94\ldots, \text{ or,}$$

$$\log_{10} e=0{\cdot}43429,44819,03251,8\ldots.$$

By finding the antilog of this, or working backwards with Table F, we obtain $e=2{\cdot}71828,1828\ldots$.

Differentiation.

If $$y=\left(1+\frac{1}{n}\right)^{nx} \quad \text{and} \quad \delta x=1/n,$$

$$\frac{\delta y}{\delta x}=\left[\left(1+\frac{1}{n}\right)^{n(x+1/n)}-\left(1+\frac{1}{n}\right)^{nx}\right]\Big/\frac{1}{n}$$

$$=\left(1+\frac{1}{n}\right)^{nx}\left[1+\frac{1}{n}-1\right]\Big/\frac{1}{n}$$

$$=y, \text{ precisely.}$$

Hence if $$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n, \quad \frac{d}{dx}(e^x)=e^x.$$

S. INMAN.

1430. You should, without fail, instruct your pupils in the six books of Euclid at least. I am, as you well know, no mathematician, and therefore my judgment in this matter is worth so much the more, because what I can do in mathematics, anybody can do; and as I can teach the first six books of Euclid, so I am sure can you. Then it is a grievous pity that at your age, and with no greater amount of work than you now have, you should make up your mind to be shut out from one great department, I might almost say, from many great departments of human knowledge. Even now I would not allow myself to say that I should never go on in mathematics, unlikely as it is at my age; yet I always think that if I were to go on a long voyage, or were in any way hindered from using many books, I should turn very eagerly to geometry, and other such studies. . . . I am quite sure that you ought not to run the risk of losing a pupil because you will not master the six books of Euclid, which, after all, are not to be despised for one's own very solace and delight; for I do not know that Pythagoras did anything strange, if he sacrificed a hecatomb when he discovered that marvellous relation between the squares containing and subtending a right angle.—Dr. Arnold in a letter to Rev. Herbert Hill in 1840, from Dean Stanley, *Life and Correspondence of Dr. Arnold* (1881), ii, p. 182. [Per Miss M. O. Stephens.]

1431. The newest verse he (" the Poet ") has no word for, and modern music sounds to him as though the composer were setting Hall and Knight's *Algebra*, or Canning's and Frere's " Loves of the Triangles ".—Thomas Burke, *Living in Bloomsbury*, 1939. [Per Mr. A. F. Mackenzie.]

THE DEFINITION OF RADIUS OF CURVATURE.

By H. T. H. Piaggio.

Several difficulties arise in the definition of the radius of curvature ρ of a plane curve. In the first place, it does not seem to have been previously noticed that the convention adopted in many textbooks for the sign of ρ leads to ridiculous consequences. Consider the very simple case of a circle of unit radius, with centre at the origin. Use A, B, C, D to denote the points $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$ respectively. Then the upper semicircle ABC is convex upwards and the lower semicircle CDA is concave upwards. G. A. Gibson, in his *Elementary Treatise on the Calculus* (p. 355), defines the sign of ρ at any point P as being that of d^2y/dx^2 at that point, or, what is equivalent, as being positive or negative according as the curve is concave upwards or convex upwards at P. From this definition it follows that $\rho=-1$ on the upper semicircle, but $\rho=+1$ on the lower semicircle! At A and C d^2y/dx^2 does not exist, so presumably ρ does not exist at these points! We cannot obtain a value by postulating continuity, since ρ is certainly discontinuous. Even the good old British principle of compromise, which suggests taking the arithmetic mean of -1 and 1, namely zero, is for once inapplicable. But there is worse to come. Let us rotate the axes, keeping the circle unchanged. Then, as the rotating axis of x passes through any point P, the value of ρ there changes suddenly from $+1$ to -1 or conversely!

The explanation of these paradoxical results is that we are trying to deal with what is naturally considered an intrinsic property of a curve by means of a definition involving not merely the curve itself, but also its position relative to arbitrary axes. Possibly this is useful when we wish to trace the curve, but, as pointed out by R. H. Fowler in his tract *The Elementary Differential Geometry of Plane Curves* (pp. 4–5), it is important to deal with invariant relations, *i.e.* those which are unchanged when we change the axes of reference. The definition of ρ as the limit, when it exists, of $ds/d\psi$, satisfies this condition. To fix the sign, we have to define arbitrarily the direction of the increase of the arc s, and also the positive direction of the axis of x. The positive direction of the tangent may then be defined as the direction in which s increases, and ψ as the angle, measured counter-clockwise, from the positive direction of the axis of x to that of the tangent. An equivalent method of fixing the sign of ρ is to define the positive normal at P as the line obtained by rotating the positive tangent through a right-angle, counter-clockwise, about P. With this set of definitions, ρ is positive or negative according as the curve and the positive normal lie on the same or the opposite side of the tangent. If we apply this to the example of the circle considered previously, $\rho=+1$ if we go round in the direction $ABCDA$, but $\rho=-1$ if we go round in the opposite direction. There is no discontinuity, and the only arbitrary part of the definition is the relation of the direction of the increase of the arc to the direction of the axes. If, however, we consider a curve made up of two or more separate parts, such as $y^2=x(x-1)(x-2)$, we can choose the direction of the increase of the arc separately for each part, so the sign of ρ on one part has no relation to the sign on the other parts.

If we wish to get rid of the arbitrary element altogether, we may do so by defining the curvature at P as the *positive* value of $d\psi/ds$ when this derivative exists, and ρ as the radius of a circle which has the same curvature as the curve has at P, so that ρ is the *positive* value of $ds/d\psi$. These are the definitions of C. J. de la Vallée Poussin in his *Cours d'Analyse Infinitésimale* (4th ed., Vol. I, p. 245), and from some points of view they are the best. The theorem that the component of acceleration along the inward normal of a point moving

along a plane curve with speed v is v^2/ρ certainly implies that ρ is positive. On the other hand, if the curvature is always to be taken as positive, we may no longer distinguish a point of inflexion from a point of undulation by saying that the curvature changes sign for the first, whereas it vanishes without changing sign for the second. Also we may no longer assert that in all cases the coordinates (X, Y) of the centre of curvature are related to (x, y), those of P, by the equations $X = x - \rho \sin \psi$, $Y = y + \rho \cos \psi$.

As a matter of fact, these equations require not only that ρ should have the sign of d^2y/dx^2, but also, very inconveniently, that ψ should be taken to lie between $-\pi/2$ and $\pi/2$, thus making ψ discontinuous as we go round a circle. This point is overlooked in some well-known textbooks, which attribute universal validity to equations based on a diagram in which both dy/dx and d^2y/dx^2 are tacitly assumed to be positive! However, Gibson (p. 355) emphasises the convention about ψ: indeed, he deduces from it his convention as to the sign of ρ. It is dangerous to use equations dependent on conventions which may easily be forgotten, and so it is better to use the alternative forms

$$X = x - \frac{dy}{dx}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \Big/ \frac{d^2y}{dx^2}, \quad Y = y + \left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \Big/ \frac{d^2y}{dx^2},$$

which are always true. The easiest way to prove them without drawing diagrams for several different cases is to write down the equation of the normal at P,

$$(Y - y)\frac{dy}{dx} + X - x = 0,$$

and, differentiating with respect to x, keeping X and Y constant, to find the point C where the normal at P meets its envelope (the evolute). To justify this on de la Vallée Poussin's scheme, first prove the theorem that the osculating circle at P has a radius equal to ρ, then define the centre of curvature C as the centre of this circle and its locus as the evolute, and finally prove the theorem that the normal at P touches the evolute at C. Another scheme, nearly the same as that given by G. Prasad in his *Text-book of Differential Calculus* (2nd ed. pp. 70–73), is to *define* the centre of curvature C as the point where the normal at P meets its envelope, and to define ρ as the positive value of the length PC. It is easy to deduce that ρ is the positive value of $ds/d\psi$ when it exists. This is perhaps the method which simplifies the mathematics as much as possible, but it is open to the objection of not being the most natural way of approaching the subject. Some consider that the most natural approach is to define ρ as the radius of the osculating circle. However, to do this we need a theorem on implicit functions giving the conditions which enable us to transform the equation of the curve from $f(x, y) = 0$ to $y = F(x)$. Some timid souls, remembering that, according to Fowler (p. 70), even so careful a writer as Goursat can go astray over certain conditions of contact, may consider it hazardous to have a fundamental definition entangled with the subtle snares of the theory of functions. Even if we restrict ourselves to algebraical curves, we have to deal with the problem of reversion of series, which contains difficulties much greater than appear on the surface. For these reasons some prefer the $ds/d\psi$ definition, in which the difficulties of the theory of functions do not seem to occur. Unfortunately this simplicity is only an illusion, for when we deal with the arc s we tacitly assume that the curve is rectifiable, which, as Jordan has shown, is equivalent to an assumption that certain functions have a property known as " bounded variation ".

The danger of tacit assumptions is especially great in the case of a cusp. A diagram suggests that a cusp may be regarded as the limiting case of a

node, and so that the arc s remains stationary while the angle ψ increases by π, giving ρ zero. This reasoning is quite unsound, for $ds/d\psi$ does not exist at the cusp. Even if we avoid this difficulty by defining ρ at a cusp P as the common limit, if it exists, of the value of ρ at points Q and R near P, one on each branch of the curve, we cannot evaluate this limit until we obtain an expansion valid near P. This expansion may take a variety of forms. Let us examine a few of the simplest. The curves $y^2=x^3$ and $y^2=x^5$ both have cusps at O with the branches on opposite sides of the tangent at O, but ρ is zero for the first curve and infinite for the second. The curve $y=ax^2 \pm bx^{5/2}$, where neither a nor b is zero, has a cusp with both branches on the same side of the tangent, and ρ comes to the positive value of $\pm 1/2a$, which may have any finite positive value. The investigation of the general case is complicated.

To sum up, no one method of defining the magnitude and sign of ρ is free from disadvantage. Personally I prefer the method of de la Vallée Poussin, but this may be due to the "halo-effect" of his high reputation for clarity. vigour, and simplicity. Some who have given deep thought to the subject prefer one or other of the alternative methods. The final choice is largely a matter of taste. H. T. H. Piaggio.

MANCHESTER BRANCH.

At the recent meeting of the Branch the following officers were elected: *President*, Miss Stephens; *Vice-Presidents*, Mr. Dakin, Miss Carver, Mr. Gregory, Professor Hartree; *Treasurer*, Miss Garver; *Secretary*, Miss Holgate; *Committee*, Miss Whittle, Mrs. Phillips; *Member on Council*, Miss Garver. F. M. Holgate, Hon. Sec.

QUEENSLAND BRANCH.

Report for 1941–1942.

The last Annual Meeting was held on 4th April, 1941. The Annual Report and Statement of Receipts and Expenditure were read and were adopted, and the Officers for the coming year were elected. The subject of Professor Simonds' Presidential Address was "Continuous Groups".

During the year two General Meetings were held, both at the University. At the first, held on 20th June, 1941, Mr. J. C. Deeney read a paper entitled "Mathematics a Social Necessity", and at the second, held on 24th October, Mr. J. P. McCarthy read a paper on "The Early Development of Mechanics".

The number of members of the Branch is at present 28, of whom 9 are members of the Mathematical Association. The financial statement shows a credit balance of £12 7s. 5d.; the increase is accounted for partly by the payment of advance subscriptions. In spite of war conditions the *Mathematical Gazette* comes to hand and is circulated.

Committee: *President*, Professor E. F. Simonds; *Vice-Presidents*, Mr. S. Stephenson, Mr. R. A. Kerr; *Hon. Secretary and Treasurer*, Mr. J. P. McCarthy; *Members*, Miss E. H. Raybould, Messrs. I. Waddle, E. W. Jones, J. C. Deeney, P. B. McGovern. J. P. McCarthy, Hon. Sec.

CORRESPONDENCE.

STARRED QUESTIONS.

To the Editor of the *Mathematical Gazette*.

Sir,—Since my letter on this subject went to press, Mr. Robson has sent me this very neat arithmetical solution of the scholarship problem: if

a, b, c, d are positive integers and $bc - ad = 1$, prove that there is no fraction x/y between a/b and c/d such that y is less than $b + d$.

If $$a/b < x/y < c/d,$$
then $$ay/b < x < cy/d,$$
and $$(ay+1)/b \leqslant x \leqslant (cy-1)/d,$$
so that $$b + d \leqslant (bc - ad)\, y = y.$$

Yours, etc.,

R. C. Lyness.

SCHOOL CERTIFICATE MATHEMATICS.

To the Editor of the *Mathematical Gazette*.

Sir,—The Local Examinations Syndicate of the University of Cambridge have just issued a new Alternative Syllabus in Geometry and Trigonometry for the School Certificate Examination, copies of which may be obtained from Syndicate Buildings, Cambridge, on application. I have been instructed by the Committee responsible for the syllabus to ask for space in your columns to call attention to some points which they have in mind.

1. The courses leading up to School Certificate Geometry have been governed for twenty years by the agreement reached in or about 1923 by the Examining Bodies to adopt a uniform list of theorems. This followed reports issued by the Mathematical Association and the Association of Assistant Masters.

2. Although the time is now ripe for a change, the Syndicate recognise that no single Examining Body can hope to devise a syllabus that will be universally accepted. They have ventured to issue an alternative syllabus now because, unless some move is made during the war to meet the demand for (1) more trigonometry, (2) less memory work, (3) more three-dimensional work, mathematical teaching may get more and more out of step with the times.

3. The Committee acknowledge with gratitude the advice they received from members of the Teaching Committee of the Mathematics Association, and offer their apologies for giving so little time for discussion. They hope that by hastening the issue of the syllabus and so making possible its experimental adoption in a few schools, they are providing concrete material for discussion among members of your Association out of which a new agreed mathematical syllabus may emerge after the war.

4. In reducing considerably the number of theorems to be learned but retaining a fixed sequence over a small field, the Committee have been influenced not only by the need to make room for more trigonometry but also by a desire to give boys and girls an opportunity of appreciating the significance of a rational argument without being wearied and perhaps confused by having to learn a large number of formal theorems whose logical sequence has become somewhat obscured. The syllabus stresses the connection between geometrical facts and suggests that the standard form for riders should be " Assuming X, prove Y ".

My own opinion, based on experience of syllabus construction over the last fifteen years and shared by others on the Committee, is that we now need a permanent but fully representative body which could review School Certificate and Higher School Certificate Mathematics syllabuses from time to time in the light of experience and the emergence of applications in new fields.

Yours faithfully,

J. L. Brereton,

(Assistant Secretary,

University of Cambridge Local Examinations Syndicate).

MATHEMATICAL NOTES

1646. *The factorising of quadratics.*

The following addition to Note No. 1536 in the *Gazette* of October 1941 may be of interest.

To factorise $63x^2 - 95x + 22$.

We require two numbers which when multiplied give 1386 (63×22) and when added give -95. Pupils set out the work, as shown, at the side of the page on which they are working. The factors are found by means of a step-by-step approach, but, with practice, learners soon see how to abbreviate the process; for instance, in the example given, the first few steps show that the approach is too slow, and the next dozen lines or so may be omitted. "Bracketing" may also be used in order to shorten the working, for instance:

$$\begin{array}{rl} & \underline{+1386} \\ -95 & = -94 - 1 \\ & = -93 - 2 \\ & = \cdots \\ & = -80 - 15 \\ & = \cdots \\ & = -77 - 18 \end{array}$$

$$-95 = -80 - 15\ ;\ 80 \times 15 = 1200, \text{ which is less than } 1386\ ;$$

$$-95 = -75 - 20\ ;\ 75 \times 20 = 1500, \text{ which is greater than } 1386\ ;$$

therefore the numbers required lie between 80×15 and 75×20.

Since $-95 = -77 - 15$, we factorise as follows:

$$\begin{aligned} & 63x^2 - 95x + 22 \\ &= 63x^2 - 77x - 18x + 22 \\ &= 7x(9x - 11) - 2(9x - 11) \\ &= (9x - 11)(7x - 2). \end{aligned}$$

Here, again, we may abbreviate by setting down, not $63x^2 - 77x$, but, since $7x$ divides both terms, $(9x - 11)$, and because $63 \div 9 = 7$ and $22 \div 11 = 2$, the other factor $(7x - 2)$ can be written down at once.

To factorise $40x^2 + 34x - 63$ (the example given in Note 1536).

We require two numbers which when multiplied give -2520 and when added give $+34$. Since the product is to be negative, *one, at least,* of the factors must be greater than 34. (If the product had been positive, *i.e.* $+2520$, *both* factors would be less than 34.)

We arrange the working as before.

As one of the factors must be a multiple of ten, a number of steps may be omitted, as shown.

$$\begin{array}{rl} & \underline{-2520} \\ +34 & = 40 - 6 \\ & = 44 - 10 \\ & = 50 - 16 \\ & = 54 - 20 \\ & = 70 - 36 \end{array}$$

This method has at least three advantages. Take, as an example, the factorising of $40x^2 + 34x - 63$.

(1) It is easier to begin with the 34 and to find two numbers of which it is the difference and of which the product is 2520, (40×63), than it is to begin by trying to find what arrangements of the prime factors, 2^3, 3^2, 5, 7, of 2520 will give two factors of which the difference is 34.

(2) Each step in the working takes one nearer to a solution, whereas in the method which begins with the factors, the failure of one arrangement of the prime factors gives little or no indication of what the next and better arrangement ought to be.

(3) It shows clearly, what the other method does not, when a quadratic has no factors with integral coefficients.

For instance, if we attempt to factorise $40x^2 - 34x - 65$, $40 \times 65 = 2600$.

Now $-34 = -70 + 36$, and $70 \times 36 = 2520$, which is less than 2600;

and $-34 = -71 + 37$, and $71 \times 37 = 2627$, which is greater than 2600.

Since the numbers required lie between the consecutive steps 70×36 and 71×37, they cannot be integers; therefore $40x^2 - 34x - 65$ has no factors with integral coefficients.

The method described above may be used to factorise quadratics of the form x^2+ax+b; indeed it is well to introduce the method by applying it to this form. To factorise $x^2-32x+247$.

$$\begin{array}{rl} & \underline{+247} \\ -32 = & -31 - 1 \\ = & -29 - 3 \\ = & \cdots \\ = & -19 - 13 \end{array}$$

Here, since the product 247 is odd, the factors must be odd; therefore the steps with even numbers are omitted.

$$x^2-32x+247=(x-19)(x-13).$$

The writer taught this method during many years and found it very satisfactory. JAS. W. STEWART.

1647. *On Note* 1536.

I was taught the following aid to factorising trinomials thirty years ago; I have never seen it in print, but I have always found that my classes have taken to it readily. We call it the Cross Method.

Consider the multiplication of $4x+7$ and $10x-9$. The processes follow the lines indicated.

$$\begin{array}{cc} 4x & +7 \\ 10x & -9 \end{array}$$

If this is written in a simpler form:

(i) vertically: $4\times 10=40$; $7\times 9=63$;

(ii) the middle term in the product is obtained by adding (or subtracting) the results of cross multiplication.

$$\begin{array}{cc} 4 & +7 \\ 10 & -9 \end{array}$$

In actual practice, if all numerical factors have first been removed:

(i) no numbers having a common multiple will lie in the same row, for example, no two even numbers;

(ii) start by factorising the number having the least number of pairs of factors, in this case 63. If 3 and 21 are tried, the various factors of 40 can be tried and rejected. Note 1608 helps here.

$$\begin{array}{cccc} \cancel{8} & \cancel{4} & \cancel{10} & 3 \\ \cancel{5} & \cancel{10} & \cancel{4} & 21 \end{array}$$

But I have always been amazed by the way in which this device enables boys to stumble quickly on the required factors.

A jingle for four-term factorisation:

"If they won't go in pairs,
Try the difference of squares." T. H. WARD HILL.

1648. *On Notes* 1594 *and* 1595 (*Vol. XXVI, pp.* 132 *and* 133).

There is so much "feeling" in both Mr. Robson's and Professor Neville's criticisms of my original note on the sign of the perpendicular from a point to a line that I cannot help suspecting that they are both merely reacting in a conservative manner to something new rather than thinking about it.

May I suggest to Professor Neville that even he cannot afford to be "obstinate"? He "obstinately asserts" that "the formation of rules for attaching a definite sign to the perpendicular . . . is mis-applied labour". The "philosophic" reasons which follow may be quite sound, but arguments by analogy are not mathematics. I think my convention for fixing the positive sense of every straight line in the plane is a useful and necessary mathematical tool, and if I err, then I do so in very good company; for I find that the necessity for doing this is dealt with by no less a mathematician than De Morgan, whose solution is, however, different from, and more complicated than, mine. I am indebted to Dr. G. J. Lidstone for the following references: (1) *Differential and Integral Calculus*, pp. 341–3; (2) *Trigonometry and Double Algebra*, pp. 7–8; (3) *The English Cyclopedia*, vol. vii, col. 503, article "Sign".

Both my critics have misunderstood my statement that "just as we have

a convention of signs for distances along OX and OY, so we must have a convention of signs for any straight line in the plane ". This does not mean that merely because we have one, therefore we must have the other; but that the same reasons which have led us to have one should lead us to have the other. I did not specify the reasons because I imagined they would be obvious to every student of " Modern Geometry ", in which lines or distances are conceived as directed entities.

Here are the reasons :

(i) It is the natural thing to do.

(ii) As soon as we start dealing with such distances by means of algebra—an algebra of directed numbers—we either make our choice of the positive sense consciously or unconsciously; for our algebra will give us a result with a definite sign. For example, in the case of the line $ax+by+c=0$, both my superior critics inform me that $ax'+by'+c$ is positive on one side of the straight line and negative on the other (like the perpendicular which is proportional to it!); but both overlook the very obvious fact that if the equation of the straight line be written $-ax-by-c=0$, which is equally good, then the side which gave a positive value will give a negative one, and vice versa. Hence, if we have no rule (such as results from my convention of signs) for choosing between $ax+by+c=0$ and $-ax-by-c=0$, we inevitably use one or other of them and get a result which has a definite sign; which means that we have unconsciously chosen the positive sense of the perpendicular from the point to the line.

I say it is better to make our choice consciously; it makes for clarity and simplicity of thought. Thus instead of having always to compare the signs of two perpendiculars—as all Mr. Robson's methods (ii), (*a*), (*b*), and (*c*) do in his example from Smith's *Conics*—when essentially we are only concerned with *a* perpendicular, my convention and rule enable us to deal with *the* perpendicular.

Mr. Robson's reasonableness may be judged by the fact that he opens by doubting whether the rule for the sign of the perpendicular which consists in comparing it with the perpendicular from the origin is the " accepted " one, but later refers to it as the " traditional " one!

He proceeds to suggest that I am one of the " short-sighted " and " foolish " schoolmasters who are merely concerned to devise mechanical (and " foolish ") rules to help schoolboys to answer tricky questions. I should like him to know that, as far as I can remember, I have never even seen an examination question of the kind described in his imaginary history, and perhaps the Editor will be good enough to help Mr. Robson out of his misconception by finding space for the enclosed short note giving a re-presentation of the whole of the book-work on perpendiculars in the light of my convention. I have used this—wisely or foolishly—with my VIth Form.

He refers to my " infallible device for lifting the unfortunate student out of the frying-pan into the fire ". What fire? And he ends by the pontifical dictum which indicates that the use of my convention complicates the solution of every problem to which it is applied. Is his example from Smith's *Conics* supposed to be a case in point? Here is the solution based on my convention, and I submit that it is clearer, shorter and simpler than that given in Smith or indicated by Mr. Robson :

" The equations of BC, CA, AB are (making the y term negative)

$$\begin{aligned} -13x-16y+453&=0 \quad \text{(i)},\\ 19x-8y-3&=0 \quad \text{(ii)},\\ x-4y+7&=0 \quad \text{(iii)}. \end{aligned}$$

From a rough plotting of the triangle, it is seen that I is below (i) and (ii). Thus the equation of CI is

$$\frac{-13x-16y+453}{\sqrt{(13^2+16^2)}}=\frac{19x-8y-3}{\sqrt{(19^2+8^2)}};$$

and that I is below (ii) and above (iii).

Thus the equation of AI is

$$\frac{19x-8y-3}{\sqrt{(19^2+8^2)}}=-\frac{x-4y+7}{\sqrt{(1^2+4^2)}};$$

whence the coordinates of I follow."

Finally, Mr. Robson shows that he himself is really only interested in solving practical problems by his naïve objection to my giving a special rule for dealing with the perpendiculars on to $x=a$! Of course I was being merely academic: as a practical problem it is always obvious whether the perpendicular is positive or negative; but theoretically the rule that $(ax'+by'+c)/\sqrt{(a^2+b^2)}$ gives the (algebraic) value of the perpendicular from $(x'y')$ to $ax+by+c=0$ ought to be applicable to a line parallel to the y-axis.

In this connection the really interesting observation would have been that it is somewhat unsatisfactory (theoretically) that there should be an exceptional case which has to have a rule all to itself. This point was made and dealt with in my note as first sent to the Editor, but was cut out because he could not find space for so long an article. The investigation leads to an interesting result, and I hope shortly to embody it in a separate note.

Outline Re-presentation of Book-work on Straight Line consequent on Convention of Signs for Lines in a Plane.

Convention.

1. Parallel lines have positive ends in same sense.
2. All directions included in range $\theta=0°$ to $180°$ (excluding the latter).
3. Sense of bounding line is the positive sense for that direction.
4. Perpendicular from point to line is therefore positive if point below line and negative if above. (See figs. 4 and 5, Note No. 1551, Vol. XXV.)

4 (a). When line is parallel to y-axis, there is no above or below; perpendicular is positive if point to left of line (and negative if to right).

Book-work.

5. (p, α) equation: α between $0°$ and $180°$, p may be negative.

The general proof by Projection holds without modification, but this is generally too difficult (or subtle) for beginners who will probably only recently have started Trigonometry. The easiest general proof is, I think, that which regards the problem as that of writing down the equation of a straight line through $(p\cos\alpha, p\sin\alpha)$ in direction $(90°+\alpha)$ where α is acute, or $(\alpha-90°)$ when α is obtuse. In both cases $m=-\cot\alpha$ and equation is

$$y-p\sin\alpha=-\cot\alpha\,(x-p\cos\alpha),$$

which reduces to $$x\cos\alpha+y\sin\alpha=p.$$

6. To transform equation of line to (p, α) form.

Ex.: $$3x-4y-10=0$$

may be written $-x\frac{3}{5}+y\frac{4}{5}=-2$ (y term positive, because $\sin\alpha$ is always positive) so that $\sin\alpha=\frac{4}{5}$, $\cos\alpha=-\frac{3}{5}$.

Thus from tables, $\alpha=143°\ 8'$, $p=-2$.

Note: y term must be made positive, and this is best done before dividing by $\sqrt{(a^2+b^2)}$, in general case.

6 (*a*). Line parallel to y-axis : $2x+3=0$ may be written

$$x\cos 0^\circ + y\sin 0^\circ = -\tfrac{3}{2}\,;\quad \alpha = 0^\circ,\quad p=-\tfrac{3}{2}.$$

Here the x-term must be made positive.

7. Perpendicular from (x', y') to $x\cos\alpha + y\sin\alpha = p$ (y term positive). Using the standard figure, and calling (x', y') P, the perpendicular PM, the perpendicular from origin ON and its intersection with parallel through P to given line R,

the parallel through $(x'y')$ is $\quad x\cos\alpha + y\sin\alpha - OR = 0,$

provided $\quad x'\cos\alpha + y'\sin\alpha - OR = 0,$

i.e. $\quad OR = x'\cos\alpha + y'\sin\alpha.$

Thus $\quad PM = RN = \vec{RO} + \vec{ON}$ (for all relative positions of R and N)

$$= ON - OR$$

$$= p - x'\cos\alpha - y'\sin\alpha.$$

Hence rule : Substitute x', y' for x, y in (p, α), equation written so that y term is *negative*.

8. Algebraic length of perpendicular from (x', y') to $ax+by+c=0$.

Write equation so that b is negative and transform to (p, α)-form by dividing by $\sqrt{(a^2+b^2)}$; then

$$\text{perpendicular} = \frac{ax'+by'+c}{\sqrt{(a^2+b^2)}} \quad (b \text{ negative}).$$

Cor. : If this comes out positive, point is below line ; if negative, above.

8 (*a*). Line parallel to y-axis : $x-a=0$.

Perpendicular (PN) = abscissa of N − abscissa of P
(for all relative positions of N and P)
$= a - x'$.

Hence rule : Write equation $-x+a=0$, or more generally $ax+b=0$ (a negative), and perpendicular $=(ax'+b)/a$.

Cor. : If this comes out positive, point is to left of line ; if negative, to right.

9. To transform $ax+by+c=0$ to form

$$\frac{x-x'}{\cos\theta} = \frac{y-y'}{\sin\theta} = r \quad (\theta \text{ between } 0 \text{ and } 180^\circ, \text{ and so } \sin\theta \text{ is positive}).$$

Case of positive gradient, *e.g.* $3x-4y+5=0$ leads without difficulty to

$$\frac{x+\frac{5}{3}}{\frac{4}{5}} = \frac{y}{\frac{3}{5}} = r.$$

Case of negative gradient, *e.g.* $3x+4y-5=0$ will similarly become

$$\frac{x+\frac{5}{3}}{-\frac{4}{5}} = \frac{y}{\frac{3}{5}} = r,$$

where, since $\sin\theta$ must be positive, the minus must be put into the cosine, *i.e.* denominator of first fraction.

It has been pointed out to me that, in my note (No. 1551, October 1941) on the sign of the perpendicular from a point to a line, the rule given at the end for distinguishing between the bisectors of the angles between two given lines is incomplete, as it does not provide for the case where one of the straight

lines is parallel to the y-axis. The completion of the rule follows from the analysis and reads as follows:

"If one of the lines is parallel to the y-axis (say $b'=0$), then make a' negative: the + sign then gives the bisector of the angle in which points are above (or below) the first line and to the right (or left) of the second, and the − sign gives the bisector of the angle in which points are above (or below) the first line and to the left (or right) of the second." C. BLACK.

1649. *The construction of hyperbolic loci.*

In Fig. 1 I shall suppose at first that the straight line OBA and the point H are given.

A series of parallel transversals, of which AH is the first, are drawn to make equal intercepts on OA, and so on OH; and a series of rays from O, of which OA is the first, are drawn to make the same number of equal intercepts on BH. Then every ray intersects the corresponding transversal on the hyperbolic arc which touches OH at O. Complete the parallelogram $OBHK$, and let AK cut OH at L. Then if LC bisects OA at C and cuts the curve at Y, YX is the maximum ordinate of the curve and AL is the tangent at A.

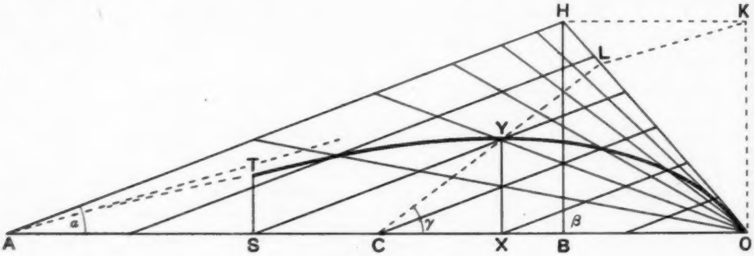


FIG. 1.

Let the angles OBH, AOH, OAH be β, ω and α respectively; and let $\cot\beta=l$, $\cot\omega=m$, $\cot\alpha=n$. Then the equation to the curve is

$$(x+ly)(x+ny)=a(x-my).$$

If $\angle LAC=\lambda$, $\angle LCO=\gamma$, then $\cot\lambda=l+m+n$, $\cot\gamma=\frac{1}{2}(l+n)$.

For practical purposes the procedure is simplified, and little elasticity is lost, if we take $\beta=90°$, $l=0$. Then the equation becomes $y=x(a-x)/(nx+ma)$.

To construct the arc, we have to obtain a, m and n from what I shall call its main offsets, namely, $OX=X$, $YX=Y$, and $ST=T$, the ordinate at $x=2X$. It will be found that

$$a=X(4Y-3T)/2(Y-T),\quad m=X^2/aY,\quad n=(a-2X)/Y.$$

The figure shows the half-deck plan of a 12-ft. racing dinghy, for which $X=72''$, $Y=27''$, $T=18''$, so that $a=216$, $m=8/9$, $n=8/3$, and the equation is

$$y=3x(216-x)/8(x+72).$$

If the origin is transferred to the point Y, the equation becomes

$$y=3x^2/8(144-x),$$

or, in its general form,

$$y=Yx^2/\{X(a-X)-(a-2X)x\}.$$

The three figures in Fig. 2 show a comparison (surely very instructive for students) of hyperbolic, parabolic and elliptic arcs, having practically the same main offsets; thus $X=3$, $Y=1$ in each case, while $T=2/3$ for the hyperbola, 0·64 for the parabola, and 0·64 almost exactly for the ellipse.

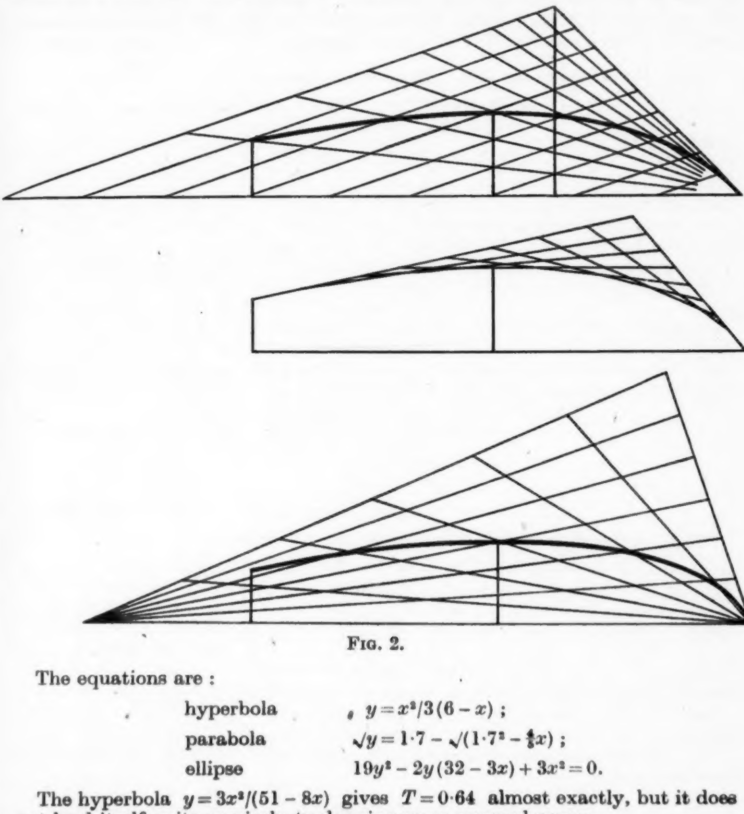


FIG. 2.

The equations are:

hyperbola	$y=x^2/3(6-x)$;
parabola	$\sqrt{y}=1\cdot 7-\sqrt{(1\cdot 7^2-\frac{4}{5}x)}$;
ellipse	$19y^2-2y(32-3x)+3x^2=0$.

The hyperbola $y=3x^2/(51-8x)$ gives $T=0\cdot 64$ almost exactly, but it does not lend itself quite so nicely to drawing upon squared paper.

It will be seen that the only important difference between these deck-plans is in the shape of the bow, where we have $dy/dx=1$, $8/7$, 3, and $26/27$ for the four equations in the order given.

I should like to ask for a construction for the envelopes corresponding to these hyperbolic loci.

A. ROMNEY GREEN.

1650. *On Note* 1514.

In the *American Mathematical Monthly* for November 1915, p. 321, I gave a cyclic dodecagon as an interesting special solution of the problem: to locate n points in the plane so that the ${}_nC_2$ distances shall be integral. To make

the figure, draw the circumcircle of an equilateral triangle whose side is 91. Then start at any corner and put in consecutive chords: 11, 39, 19, 39; 11, 39, 19, 39; 11, 39, 19, 39. The following numbers will appear as diagonals: 49, 56, 65, 80, 85, 91, 96, 99, 104, 105. They can be readily sorted into their places by anyone who has the patience. The fact that Ptolemy's theorem can be verified in at least 81 different ways entitles the set of numbers to rank with the most dignified of magic squares. NORMAN ANNING.

1651. *Properties of the sum of the integer powers.*

Let
$$g_r = 1^r + 2^r + \dots + n^r,$$
$$f_r = \tfrac{1}{2}\{(n+1)^r - n^r - (2n+1)\}.$$

1°.
$$g_{2r} = \frac{(-)^{r+1}2^{r+1}}{(2r+2)!}\begin{vmatrix} \binom{2r+2}{4} & \binom{2r+2}{6} & \cdots & \binom{2r+2}{2r} & f_{2r+2} \\ \binom{2r}{2} & \binom{2r}{4} & \cdots & \binom{2r}{2r-2} & f_{2r} \\ 0 & \binom{2r-2}{2} & \cdots & \binom{2r-2}{2r-4} & f_{2r-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \binom{4}{2} & f_4 \end{vmatrix}$$

$$g_{2r-1} = \frac{(-)^{r+1}2^{r}}{(2r+1)!}\begin{vmatrix} \binom{2r+1}{4} & \binom{2r+1}{6} & \cdots & \binom{2r+1}{2r} & (f_{2r+1}+n) \\ \binom{2r-1}{2} & \binom{2r-1}{4} & \cdots & \binom{2r-1}{2r-2} & (f_{2r-1}+n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \binom{3}{2} & (f_3+n) \end{vmatrix}$$

2°. g_{2r} is divisible by $n(n+1)(2n+1)$, for $r \geqslant 1$,
g_{2r+1} is divisible by $n^2(n+1)^2$, for $r \geqslant 1$.

3°. $g_r/r! = n^{r+1}/(r+1)! + \frac{1}{2}n^r/r! + \frac{1}{12}n^{r-1}/(r-1)!$

$$-\begin{vmatrix} \frac{1}{3!} & \frac{3}{2\,.\,5!} \\ 1 & \frac{1}{2\,.\,3!} \end{vmatrix}\frac{n^{r-3}}{(r-3)!} + \begin{vmatrix} \frac{1}{3!} & \frac{1}{5!} & \frac{5}{2\,.\,7!} \\ 1 & \frac{1}{2!} & \frac{3}{2\,.\,5!} \\ 0 & 1 & \frac{1}{2\,.\,3!} \end{vmatrix}\frac{n^{r-5}}{(r-5)!} - \begin{vmatrix} \frac{1}{3!} & \frac{1}{5!} & \frac{1}{7!} & \frac{7}{2\,.\,9!} \\ 1 & \frac{1}{3!} & \frac{1}{5!} & \frac{5}{2\,.\,7!} \\ 0 & 1 & \frac{1}{3!} & \frac{3}{2\,.\,5!} \\ 0 & 0 & 1 & \frac{1}{2\,.\,3!} \end{vmatrix}\frac{n^{r-7}}{(r-7)!} + \dots$$

If r is even, the last term is

$$(-)^{(r+2)/2}n\begin{vmatrix} \frac{1}{3!} & \frac{1}{5!} & \cdots & \frac{1}{(r-1)!} & \frac{r-1}{2(r+1)!} \\ 1 & \frac{1}{3!} & \cdots & \frac{1}{(r-3)!} & \frac{r-3}{2(r-1)!} \\ 0 & 1 & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & 1 & \frac{1}{2\,.\,3!} \end{vmatrix};$$

if r is odd, the last term is

$$(-)^{(r+1)/2} n^2 \begin{vmatrix} \frac{1}{3!} & \frac{1}{5!} & \cdots & \frac{1}{(r-2)!} & \frac{r-2}{2\,.\,r!} \\ 1 & \frac{1}{3!} & \cdots & \frac{1}{(r-4)!} & \frac{r-4}{2(r-2)!} \\ 0 & 1 & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & & 1 & \frac{1}{2\,.\,3!} \end{vmatrix}.$$

J. STORR-BEST.

1652. *Simple subtraction.*

There are several methods in common use for performing simple subtraction. This note is not concerned with the respective merits of the various methods, though I would point out that Dr. P. B. Ballard and others have shown conclusively that the method of " equal additions " produces better results than the method of " decomposition ". The main objection to the " equal addition " method has been the difficulty of explaining it.

The explanation given below seems to me to clear away all this difficulty. It was suggested to me by a very young boy some thirty years ago, and appears in Godfrey and Siddons' *Teaching of Elementary Mathematics*; but I still see arithmetic books giving elaborate explanations of the method that must be difficult, if not unintelligible, to the majority of children.

The particular example the small boy was working was

From 94
Take 58

He had already learnt, with matches, that

$$9 \text{ tens} + 4 \text{ units} = 8 \text{ tens} + 14 \text{ units},$$

i.e. he could do the sum by the " decomposition " method.

I then tried to explain to him that (having said 8 from 14 leaves 6), instead of saying 5 from 8 leaves 3, he could say 6 from 9 leaves 3. He thought for a moment and then said, " Oh, I see, you take the 1 and the 5 away (from the 9) *at the same time.*"

When I showed this explanation to Dr. Ballard, he wrote: " The small boy's explanation of ' borrowing ' shows that flash of illumination which charms one immensely, in the young mind, or indeed in any mind. I don't speak from the ' superior ' standpoint, for I found the small boy's remark illuminating to myself: I had never thought of it in that light before."

A. W. SIDDONS.

1653. *On Multiplication of Money.*

This note is in reply to Messrs. Siddons and Webb's Note 1630, *Math. Gazette*, Dec. 1942, p. 216. Unfortunately I was deflected from my study of this topic over a year ago before completing it, and I am rather rusty and not completely determined about it. However, here are my comments on the points raised:

P. 216. (i) I think the best ultimate method is a mixed one in which the ingredients depend on the numbers involved and the ability of the pupil. It would often be a purely practice method. But I would cut out the extravagances of Practice as I would those of the Box Method. Thus in multiplying 14s. 5d. I would treat 5d. as 10d. $\div$ 2, and not 4s. $\div$ 12 + $\frac{1}{4}$ of 4d. (or 1s. $\div$ 12,

which is better). Similarly, 3s. as a whole. My general position is that certain exceptions to pure practice methods should be standardised for use in the ultimate method. These would vary with the calibre of the pupils. I would avoid formal Practice in the case of weak pupils, but would introduce some fractional work into the Box Method. The formal setting out of Practice might be abandoned in favour of the Box arrangement, for all pupils, but I had not reached a definite conclusion on this point, and I should like other opinion.

E.g.	£2	14s.		5d.		
				× 137		
	£2	£$\frac{1}{2}$	£$\frac{1}{5}$	5d.		
	£274	0	0			
	68	10	0		or here,	
685d.	27	8	0			685d.
57s. 1d. =	2	17	1			57s. 1d.

P. 216. (ii) Practice is found difficult largely because the pupil is generally plunged into it without adequate preparation. He should certainly have a simple mechanical method at the outset, but this, the Box Method, should grow into the ultimate method as the pupil's knowledge of fractions develops. This would prevent wasted effort (*e.g.* 6d. × 38 = 228d. = 19s.) in the mechanical method, help to make fractions more real, bridge the gap between the mechanical and ultimate methods, and so reduce the difficulty of Practice.

P. 217. (i) This matter should now be clear. I was writing strictly to the title of my note, and had no intention of suggesting a best method for 1s. 3½d. × 539. There was no point in introducing a multiplicand such as, say, £2 17s. 3½d., instead of 1s. 3½d., since my argument was based on the pence. I approve the ultimate method given on p. 217, except for bright pupils capable of treating 1s. 3d. as £$\frac{1}{16}$.

P. 217. (ii) I hope I may differ with regard to the word " arrangement " which does not seem to me apposite enough. There is a definite difference in method of procedure, which, in fact, Mr. Siddons points out as a distictive feature of his procedure, or version of the Box Method, *Math. Gazette*, Feb. 1941, p. 36 (ii). I may add that this statement was the basis of my opinion that Mr. Siddons' method was not adaptable to the fractional method, This is still my opinion, unless the advantage, a valuable one, of Mr. Siddons' method is to be impaired.

Would it not be easy to make, *e.g.* 6d. × n = ½s. × n = etc., mechanical at an early age?

R. S. WILLIAMSON.

1654. *Cross multiplication in arithmetic.*

In a review in the *Gazette*, July 1941, p. 197, H. W. gives a method of multiplying two numbers in one line by inverting the multiplier. This method is not new, but is just a modified form of the historic method of Cross Multiplication discussed by D. E. Smith in his *History of Mathematics*, II, 112. I have used Cross Multiplication from early youth when I learnt the method from Sol Stone, Barnum's lightning calculator. It is too confusing for the average pupil except in simple cases.

In the *Gazette* (XIII, p. 367 ; XIV, p. 193 ; *see also* XIV, p. 195) Professor Lodge gave a short method of multiplying by any number between 10 and 20. For example, in 5678 × 19 the successive figures in the product are obtained thus : 8 × 9 ; 7 × 9 + 8 ; 6 × 9 + 7 ; 5 × 9 + 6 ; 5. This is a special case, and a very useful one, of Cross Multiplication. I find that it is given in Wingate's *Arithmetic*, 1713, in the supplement by Shelley, who extends

the rule to multipliers between 20 and 30 " by taking in the Double of the back Figure ".

Cross Multiplication is fairly easy also when both multiplicand and multiplier contain two figures only, for example, 67×48. But pupils find difficulty in calculating the ten's figure of the product, thus: $(7 \times 4 + 5) + 6 \times 8$. Sang, who recommends Cross Multiplication in his *Elementary Arithmetic*, 1866, advises the student that " in general, he will find it easier to take the two cross products first ", that is, *before* finding the unit figure of the product. This certainly seems to be the case. I have not seen the suggestion before, and some teachers may be glad to hear of it.

Barnard Smith (*Arithmetic for School*, 1877, p. 167) applies the term " Cross Multiplication " to the method of Duodecimals, which is hardly used nowadays despite its utility. Perhaps this is due to the difficulty of units. I notice that Shelley, who uses the method in Wingate's *Arithmetic*, always writes down the product with incorrect units; for example, 104 ft. 6 in. × 12 ft. 3 in. is given as 142 yar. 2 feet 1 inch 6 part. His first example, 4 Foot 5. by 2 Foot 42., is curious: $2{,}42 \times 4{,}5 = 10{,}890$. Answer—10 f. 890.

R. S. WILLIAMSON.

1655. *Duodecimals in* 1842.

Extracted from the *Minutes of the Committee of Council on Education*, 1842–3 (now the Board of Education):

Page 729:

	£.	s.	d.
Gas, 21. 0s. 7d.; coals, 51. 14s. $4\frac{1}{2}$d. - - - - -	7	14	$11^{6}/$

Page 728:

By surplus 1836 - -	2,983	3	$10^{4}/$
,, 1837 - -	3,099	9	$9^{7}/$
,, 1838 - -	2,205	7	$8^{7}/$
,, 1839 - -	2,885	13	$3^{1}/$
,, 1840 - -	3,355	1	$2^{11}/$
,, 1841 - -	2,640	0	$2^{6}/$
Balance due by schools -	3,065	17	2
	£20,234	13	3

Page 730:

Painter's work, 121. 14s. 8d.; plasterer, 21. 6s. 1^{6}d. (*sic*)	15	0	$9^{6}/$
" The School Fund, etc. - - - - - -	1784	11	$1\frac{4}{12}$ (*sic*)

The above extracts refer to the Chief Schools in the Presbytery of Edinburgh. Do they reflect a custom then prevailing in Great Britain, or a well-known Scottish habit?

R. S. WILLIAMSON.

1656. *Note on the Rhind Papyrus.*

According to Peet's translation of this papyrus, page 121, ed. 1923, the scribe Ahmose (Ahmes) gives two solutions of the problem:

" Seven houses; in each are 7 cats; each cat kills 7 mice; each mouse would have eaten 7 ears of spelt; each ear of spelt will produce 7 *hekat*. What is the total of all these? "

The first solution comes by adding 7, 49, 343, 2401, 16807. The second solution gives 2801×7, with no explanation how 2801 is arrived at.

Peet shows that the product 2801×7 may be obtained by substituting $a = 7$, $r = 7$, in the formula for the sum of a geometric series, and concludes that " the solution of even this limited type of geometric series is very flatter-

ing to their mathematical intelligence ". But this product also comes from considering the total for one house first. It seems more likely that the practically-minded Egyptian found his second solution thus,

$$(1+7+49+343+2401)\times 7,$$

obtaining the last four figures in the bracket from his first solution.

I have followed Peet in describing the solutions as first and second. The order in the papyrus is the reverse. D. E. Smith, *History of Mathematics*, p. 500, gives the two solutions in papyrus order, but his translation should be corrected from Peet, and his comments read in the light of my suggestion.

R. S. WILLIAMSON.

1657. *A formula for the Reversion of a power Series* (see Note 1587).

In Note 1587, Mr. W. G. Bickley quotes the coefficients of a series approximating to the roots of an equation. I would like to record a convenient statement for the root, of an equation, nearest the origin. The root of the equation $\phi(z) \equiv b+b_1z+b_2z^2+b_3z^3+\ldots=0$ where $b_1 \neq 0$, nearest the origin, is the coefficient of $1/t$ in the expansion of $-\log[\phi(t)/t]$ in an ascending series of powers of t.

A proof may be derived from Burmann's Theorem* or may be given as follows:

If γ is a contour (in the complex plane) containing the origin and the point z and such that $\phi(t)-\phi(\zeta)$ has only one simple zero within γ, viz. at $t=\zeta$, then $\phi'(\zeta)/[\phi(t)-\phi(\zeta)]$ has, within γ, but one simple pole with residue 1. Hence, if z is the required root,

$$\begin{aligned} z=\int_0^z d\zeta &= \frac{1}{2\pi i}\int_0^z\int_\gamma \frac{\phi'(\zeta)}{\phi(t)-\phi(\zeta)}\,dt\,d\zeta \\ &= \frac{-1}{2\pi i}\int_\gamma \Big[\log\{\phi(t)-\phi(\zeta)\}dt\Big]_0^z \\ &= \frac{-1}{2\pi i}\int_\gamma [\log\phi(t)-\log\{\phi(t)-b\}]\,dt \\ &= \frac{-1}{2\pi i}\int_\gamma \log[\phi(t)/t]\,dt+\frac{1}{2\pi i}\int_\gamma \log\left[\frac{\phi(t)-b}{t}\right]dt. \end{aligned}$$

The last term vanishes, for since $b_1 \neq 0$, its argument returns to its initial value after travelling around the contour, and hence

$$z=\text{coefficient of } 1/t \text{ in the expansion of } -\log[\phi(t)/t].$$

As an example consider the cubic $x=k(1-x^3)$ where $|k|^3 \leqslant 4/27$. Then x is the coefficient of $1/t$ in the expansion of

$$-\log\left[1-\frac{k}{t}(1-t^3)\right],$$

i.e. in the expansion

$$\frac{k}{t}(1-t^3)+\frac{1}{2}\left(\frac{k}{t}\right)^2(1-t^3)^2+\frac{1}{3}\left(\frac{k}{t}\right)^3(1-t^3)^3+\ldots.$$

Hence $$x=k-\frac{1}{4}\cdot\frac{4}{1}k^4+\frac{1}{7}\cdot\frac{7\,.\,6}{1\,.\,2}k^7-\frac{1}{10}\cdot\frac{10\,.\,9\,.\,8}{1\,.\,2\,.\,3}k^{10}+\ldots$$

$$=k\left[1-k^3+\frac{6}{2}\cdot k^6-\frac{9\,.\,8}{2\,.\,3}k^9+\ldots\right].$$

W. R. ANDRESS.

* Whittaker and Watson, *Modern analysis*, 1915, pp. 129–130.

1658. *On Note* 1502.

Mr. Brewster inquired if there was any reasonably simple method of calculating $\int_0^a x^x dx$ for positive a. In the following method I express the integral as a series which for small a is fairly rapidly convergent and provides a practical method of calculating the integral.

$$I=\int_0^a x^x\,dx=\int_0^a e^{x\log x}\,dx=\int_0^a \sum_{n=0}^{\infty}\frac{(x\log x)^n}{n!}\,dx$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^a (x\log x)^n\,dx$$

(assuming that term by term integration is valid)

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^{n+1}}\int_{(n+1)\log 1/a}^{\infty} e^{-u}u^n\,du.$$

When $a=1$,

$$I=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^{n+1}n!}\Gamma(n+1)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^{n+1}}=0{\cdot}783430\ldots.$$

For $a\neq 1$, consider

$$\int_\alpha^\infty e^{-u}u^n\,du=\int_0^\infty-\int_0^\alpha e^{-u}u^n\,du=n!-\int_0^\alpha e^{-u}u^n\,du.$$

But $$\int_0^\alpha e^{-u}u^n\,du=\Big[-e^{-u}u^n\Big]_0^\alpha+n\int_0^\alpha e^{-u}u^{n-1}\,du$$

$$=-e^{-\alpha}(\alpha^n+n\alpha^{n-1}+\ldots+n!)+n!$$

by successive integration by parts ; hence

$$\int_\alpha^\infty e^{-u}u^n\,du=e^{-\alpha}n!\sum_{m=0}^{n}\frac{\alpha^m}{m!}.$$

Hence $$I=\sum_{n=0}^{\infty}I_n \text{ where } I_n=\frac{(-1)^n a^{n+1}}{(n+1)^{n+1}}\sum_{m=0}^{n}\frac{\{(n+1)\log 1/a\}^m}{m!}.$$

For $a\leqslant 1/e$ the series is rapidly convergent.

$$|I_n|<\frac{a^{n+1}}{(n+1)^{n+1}}(n+1)\frac{\{(n+1)\log 1/a\}^n}{n!}=\frac{a(a\log 1/a)^n}{n!}\leqslant\frac{a}{e^n n!}.$$

Thus for practical purposes it would be sufficient to consider the first few terms.

For $1/e<a<1$, the convergence though not so rapid as before is rapid enough to make the method practical.

$$|I_5|<{\cdot}0002a,\quad |I_6|<{\cdot}00002a.$$

So it is sufficient to consider terms up to $n=4$ or 5 according to the degree of accuracy required. For $a>1$ the series is valid, but the convergence is not rapid enough. Till a is about 1·6, $|I_5|<{\cdot}00006a$ and the method is practicable. But as a increases we should consider more and more terms ; and for large a the method does not seem useful in practice.

C. JAYARATNAM ELIEZER.

1659. *On Note* 1577.

On seeing Mr. Ramsey's article in the *Gazette* (July 1941, p. 141) on problems with changing mass I deduced the formula, which R. N. gives in the note,

and suggested it to Mr. Ramsey. But he pointed out, and I am inclined to feel with him, that with such problems it is the *method* and not the *formula* which is important and fundamental; and students would have a clearer grasp of these problems if more prominence is attached to the method than is likely to be if such a formula was adopted. It is essential to make the students realise that the mathematics in applied mathematics is only a tool and that the physical ideas and principles should be kept to the forefront as much as possible. A formula is useful mainly as an aid to memory and this particular one does not seem to me to be very helpful.

C. Jayaratnam Eliezer.

1660. *On Note* 1578.

There was a slight misprint in this note by Dr. Roth on a new and elegant method of solving the linear differential equation

$$y''+Py'+Qy=R. \qquad (1)$$

When substituting $y'=\mu y+\nu$ we should obtain

$$(\mu'+\mu^2+P\mu+Q)y+\nu'+(\mu+P)\nu=R,$$

thus leading to

$$\mu'+\mu^2+P\mu+Q=0, \qquad (2)$$

$$\nu'+(\mu+P)\nu-R=0, \qquad (3)$$

whereas, in the Note, (3) was given as $\nu'+\mu\nu-R=0$.

It is interesting to note that there is a simple and direct connection between this method and the well-known method of first finding a particular u satisfying the equation

$$u''+Pu'+Qu=0 \qquad (4)$$

and substituting $y=uz$ in (1), thus obtaining

$$uz''+(uP+2u')z'-R=0. \qquad (5)$$

If we put $u'/u=\mu$, then $u''/u=\mu'+\mu^2$ and (4) reduces to (2). Again, if we put $uz'=\nu$, then (5) reduces to (3). Thus the two methods have much in common. In fact in the second method when we proceed to find u from (4) we are led to the same Riccati equation (2). For (4) can be written

$$\{D-\alpha(x)\}\{D-\mu(x)\}u=0$$

where

$$\alpha+\mu=-P, \quad \alpha\mu-\mu'=Q;$$

hence μ is given by

$$\mu'+(P+\mu)\mu+Q=0,$$

which is the same as (2). After finding a particular μ we obtain u by solving $\{D-\mu(x)\}u=0$, that is, $u'-\mu u=0$, which is only to be expected as we had $\mu=u'/u$ earlier.

C. Jayaratnam Eliezer.

1661. *Motion with changing mass.*

In the *Gazette* for February, 1942, I drew attention to the following equation for the motion of a body with variable mass:

$$\frac{d}{dt}(M\mathbf{V})-\mathbf{u}\frac{dM}{dt}=\mathbf{F}. \qquad (1)$$

Here M is the mass of the body at time t, $\mathbf{V}$ its velocity and $\mathbf{F}$ the force acting. For simple problems involving only one kind of absorption or ejection process, $\mathbf{u}$ is the velocity of the matter at the instant of absorption or ejection. For more general problems involving more than one kind of absorption process, $\mathbf{u}$ was defined to be the velocity at time t of the mass centre of those particles absorbed in time δt. A similar definition was given for the case of more than one ejection process.

Equation (1) was established in the first place for the case in which no external forces acted upon the matter ejected from the body after the instant of ejection, or on matter absorbed by the body before the instant of absorption. The phrase " general validity " that Mr. Ramsey has questioned in his article in the *Gazette* for October, 1942, occurred in the extension of equation (1) to cases for which external forces were acting on matter before absorption or after ejection. In view of Mr. Ramsey's remarks it was perhaps misleading to refer to this equation as " *the* general equation ", and it is necessary to indicate its range of validity. The equation was established by the application of the ordinary principles of Newtonian mechanics, and should obviously be valid only for that class of problem for which the symbols M, $\mathbf{V}$, $\mathbf{u}$, $\mathbf{F}$ have meaning. That is, equation (1) should be valid for problems in which mass is changing continuously by absorption only, or by ejection only. The examples considered in Mr. Ramsey's second note, and problem (iii) of his first note, involve the *simultaneous* absorption and ejection of matter. For such problems the velocity $\mathbf{u}$ has not been defined, and clearly it is not possible to make immediate application of equation (1). The question of such application could hardly have arisen had I not misapplied the principles of chemistry in discussing Mr. Ramsey's problem (iii), by neglecting the oxygen, thereby converting the problem to one involving the ejection of matter only.

It may be noted that equation (1) may be rewritten in the form of Mr. Ramsey's equation (5), for if matter is absorbed at rates $m_{a,1}, \ldots, m_{a,n}$ with velocities $\mathbf{v}_{a,1}, \ldots, \mathbf{v}_{a,n}$, then

$$\mathbf{u} = \Sigma m_a \mathbf{v}_a / \Sigma m_a \quad \text{and} \quad \Sigma m_a = dM/dt;$$

for ejection

$$\mathbf{u} = \Sigma m_e \mathbf{v}_e / \Sigma m_e \quad \text{and} \quad \Sigma m_e = -dM/dt.$$

Equation (1) could be extended to the case of simultaneous absorption and ejection by considering an ejection process as absorption of negative mass, but such a procedure would break down when $\Sigma m_a = \Sigma m_e$.

There will be general agreement with Mr. Ramsey's statement that ability to quote general equations is a poor substitute for ability to apply general principles, but nevertheless I think that the general equations considered are not without interest. It is not entirely without importance to know that the equation $M\, d\mathbf{V}/dt = \mathbf{F}$ is valid when the velocity of matter immediately before absorption, and after ejection, is equal to the instantaneous velocity of the body, and that $d(M\mathbf{V})/dt = \mathbf{F}$ when the matter is at rest immediately before absorption, and after ejection. These conclusions were obtained from equation (1) for a body whose mass was changing by absorption processes only, or by ejection processes only. Mr. Ramsey's equation (5) shows that they remain valid when absorption and ejection processes take place simultaneously.

R. N.

1662. *Sum of the reciprocals of zero-free integers.*

The sum $\Sigma(1/n)$ where n ranges over all positive integers except those which contain a specified digit is known to converge,* but I have not heard of any calculation of the sum to infinity. The sum is greatest when zero is the missing digit. I find 23·1035... for this sum ; for any other missing digit it will not be very different. The method used is indicated below.

Put S_r for the sum of the reciprocals of all r-digit numbers excluding those which contain a zero digit, Q_r for the sum of the squares of the reciprocals of the r-digit zero-free numbers, C_r for the sum of the cubes. Since

$$\sum_{x=1}^{9} \frac{1}{10n+x} = \frac{9}{10}\cdot\frac{1}{n} - \frac{9}{20}\cdot\frac{1}{n^2} + \frac{57}{200}\cdot\frac{1}{n^3} - \ldots,$$

* Hardy & Wright, *Theory of Numbers*, p. 119.

we have $$S_{r+1}=\tfrac{9}{10}\cdot S_r-\tfrac{9}{20}Q_r+\tfrac{57}{200}C_r-\dots .$$

Summing for $r=3$ to ∞,

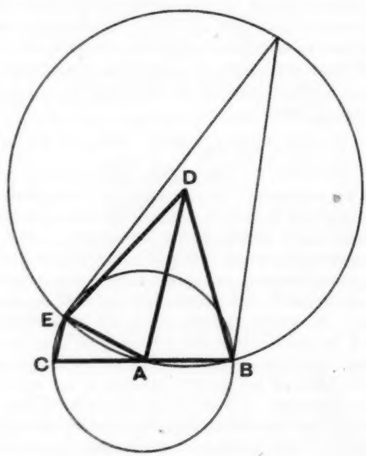
$$\sum_{r=4}^{\infty} S_r=9S_3-\tfrac{9}{2}\sum_{3}^{\infty} Q_r+\tfrac{57}{20}\sum_{3}^{\infty} C_r-\dots .$$

A similar method can be used for ΣQ_r, etc. The sums S_1, S_2, S_3, Q_1, etc., are found by the Euler-Maclaurin summation formula.

I should be much interested to hear from anyone who has worked out this or other cases, or can suggest a simpler method. G. W. BREWSTER.

1663. *On a compass construction.*

The writer of Note 1613 (XXV, p. 170) and others may be interested in the following solution of the problem of bisecting a given line using compasses only. It was set as a question in the Elementary Mathematics papers in July 1921 for the Oxford and Cambridge School Certificate.



AB is the given line. With centre A and radius AB draw a circle. By stepping off three times from B with radius AB find C, the point of the circle which is diametrically opposite to B. With centres A and B and radius BC draw arcs to cut at D. With centre D and radius DA draw the circle through A, B to cut the first circle again at E. Then $CE=\frac{1}{2}AB$. For : the angle EAC is equal to the angle subtended by the arc EAB at any point of the remainder of this circle and so

$$\angle EAC=\tfrac{1}{2}\angle EDB, \text{ at the centre}$$
$$=\angle ADB.$$

Hence the isosceles triangles EAC, ADB are similar and

$$CE/CA=AB/AD=\tfrac{1}{2}.$$

E. V. SMITH.

REVIEWS.

The Beginning and End of the World. By E. T. WHITTAKER. Pp. 64. 2s. 6d. 1942. Riddell Memorial Lectures, University of Durham. (Oxford University Press)

Before discussing this series of lectures in detail the reviewer wishes to say that they are extremely interesting, expressed with the utmost clarity and simplicity, and up to that high standard which Professor Whittaker always sets himself. Any disagreement is a sign that the lectures provoke thought.

The aim of the book is to give in clear outline those general conclusions of modern natural philosophy which bear on the doctrines of a special creation and of a Power outside our physical universe. It is valuable to have in such small compass and in close logical relationship many of the more important and recent findings of astronomical and physical science. With these there can be no quarrel, but certain arguments appear which unfortunately are popular with many mathematical physicists when writing for the general public and which ought, in the reviewer's opinion, to be marked as fallacious.

The essential point is that the laws of science are not to be accepted as absolutely true. The aim of the scientific method is not only to doubt but to investigate most carefully just those weak hinges where hypotheses in different fields of study are inconsistent with one another. Scientific laws are not dogmas, and depend on experience which is local in space and time. In particular the inconsistency between the laws of increase of entropy and of biological evolution is a sign of our present ignorance. When larger intervals of time come to be better known, either or both may be found to be untrue. The history of science suggests that what we know now is only a narrow fringe of what will be known a thousand years hence.

On p. 42 it is said of copepods in the sea that " we are struck by the fact that with them the individual counts for nothing, the race is everything ". Surely the fact is that it is we who count copepods only in the mass. What the copepod thinks we do not know, but it acts as if it were intent on saving its individual life and producing its individual offspring. It does not say to itself " The death rate has gone up so I must be particularly careful to avoid that whale tomorrow."

On p. 45 it is asserted that a certain distribution of numbers must act for all time because it does so now.

This kind of logical flaw which assumes that we are more than pickers of shells on the shore, that what happens to have been observed up to the present time is so certainly true as to be relevant to the larger domain of religion may do harm both to science and religion. But it is when we come to arguments about order and disorder that more serious flaws are found. It is true that entropy can be defined and measured but who has seen a rational definition of disorder in a finite collection? Three cards can be shuffled into six different orders. Each time a pack of 52 is shuffled there is a new order. If we start with a million packs all initially in the same order and shuffle them *independently*, their orders will be different but in an actual case the shuffling will be subject to some law, and these new orders will be again related in a vast polyphonic super-order such as Bach might enjoy.

The whole notion rests on an incorrect equation of different logical types. In probability we deal with infinite classes of events. Independence, for example in shuffling, is defined logically only for such classes. When we speak of shuffling we intend not one particular sequence of events but infinitely many possible sequences. The theory of probability, like euclidean geometry, can be made a convenient basis for the study of bodies. Each is

a self-consistent logical system. Neither is immediately open to experience. Each is useful up to a point, but when it is no longer useful we discard it. If theories of probability appear to suggest results which contradict experience surely that does not surprise us. No man-made law, not even a statistical one, is perfect. Either entropy and disorder are not so closely related to one another as modern mathematical physics supposes or else what we at present see as disorder is, like a musical discord, resolved in a higher order. To avoid confusion of argument we note that the modern theory of "disorder" in metals is quite another matter. There "disorder" means "difference of order", that is, a break in lattice pattern, a very special and simple order.

It surprises one to find also what appears to be that old and fallacious argument that if entropy always increases it must have been zero at some finite point of past time and must reach its maximum after a finite interval. But the whole question of finiteness has become difficult nowadays, when two scales of time are contemplated, one a linear function of the logarithm of the other. This particular speculation is not mentioned in the book.

Finally, the reviewer ventures to predict that when biology and psychology have grown, they will have many corrections to make to our ideas of order and disorder and of the real structure of the universe. This is, in part, what Whitehead is preaching to our reluctant ears. P. J. D.

The Non-Singular Cubic Surfaces. A new method of investigation with special reference to questions of reality. By B. SEGRE. Pp. xi, 180. 15s. 1942. (Oxford University Press)

This is an extremely elegant book. In it the manifold symmetries of the cubic surface are displayed with the utmost clarity; the new method of investigation which Professor Segre has discovered considerably simplifies the task of finding and proving them and that method has been skilfully employed to add many interesting new results to our knowledge of the cubic surface, particularly of the real cubic surface.

But admiration for the geometrical skill thus displayed only increases the bitterness of the reviewer's disappointment that Professor Segre, whose work in other spheres has shown him to be one of the greatest algebraic geometers of our time, should have spent his energies on such a subject. The reviewer feels that a mathematical investigation should satisfy one of two criteria: either it should be of some practical use; or it should concern the whole or a part of a general field of relations which will give us a new insight into the "connectedness of things", thereby increasing the *scope*, not merely the quantity, of human knowledge and providing also the hope (for it can be no more than a hope) that the new sphere of knowledge may ultimately benefit mankind in general.

Investigation of the cubic surface satisfied the second of these two criteria during the latter half of the nineteenth century. It was a useful, probably essential, step towards understanding algebraic surfaces in general. But that period in the history of geometry is now passed. Algebraic geometry, it is generally recognised, has to-day reached a crisis, through which it can pass apparently only by the introduction of radically new methods, probably of great generality. Professor Segre stands in the forefront of those who are trying to discover such new methods and he is one of the most likely to be successful. But that, alas, only adds to the reviewer's disappointment that he should spend his time in an exhibition of technical skill on such a subject as the cubic surface, which gives little promise of yielding these new methods.

The cubic surface remains important to-day in one respect—in the training of young geometers, for it is a subject on which they can conveniently practice

their technique and at the same time meet in simple form concepts whose more general significance they must later learn to appreciate. Professor Segre's book will be useful here. Almost all of it lies beyond the scope of most university degree courses, but the post-graduate student, though he will seldom want it all, will certainly improve his technical skill by reading large parts of it. S. L.

The Life of Sir J. J. Thomson, O.M. By LORD RAYLEIGH. Pp. x, 299. 18s. 1942. (Cambridge University Press)

This is an excellent biography. It is not a three-volumed sarcophagus of the kind in which so many great Victorians were interred ; in fact, it is too short, not because of any defect in style or important omissions in matter, but because we end it feeling that we would gladly hear more.

In speaking of his undergraduate days and contemporaries, Professor Forsyth said of J. J. : " His personality stood out : we felt that he was framed in an intellectual mould different from ours " (*Gazette*, XIX, p. 165). This was perhaps true of J. J. throughout his life ; certainly an undergraduate going up to Cambridge soon after the last war, knowing that there he might see and hear the great men whose names he already knew—Baker, Eddington, Hardy, Hobson, Larmor, Littlewood, Rutherford—still felt that J. J.'s mighty stature o'ertopped them all. And Rayleigh's book shows clearly that this impression held by a humble undergraduate was held also by all the great physicists of the era. If the book does not entirely explain the source of this feeling, it does show how universal it was. There is no trace of Boswellism in the writing ; criticism is discerning in both praise and very occasional blame. But throughout it J. J. stands out clearly as a great physicist, a great Master of Trinity, and a great man—perhaps greater in some ways than those who did not know him personally had realised.

There are many good stories in these pages, not dragged in as casual ornaments, but woven into the text to illustrate its theme. The use of quotations is particularly happy ; there is a most interesting account, from Rouse Ball, of Thomson's installation as Master of Trinity, and a vivid cameo by Aston from *The Times*, illustrating an " intuitive ability to comprehend the inner working of intricate apparatus without the trouble of handling it . . . verging on the miraculous, the hall-mark of a great genius ". And even a brief review may take note of the two elderly ladies, one who in 1915 was " much upset by a bad smell and thinks it might be bottled up and used against the Germans ", and the other who wrote in 1930 to ask J. J. if he thought she " would be safe in putting a substantial bet on Cambridge " for that year's Boat Race. The photographs are admirable, including a strikingly realistic one of J. J. with Dr. Irving Langmuir in the Laboratory of the General Electric Company, Schenectady, and a happy one of breakfast at the Lodge, Trinity, taken by the author.

Of course the great theme of the book is J. J.'s work at the Cavendish. Fortunately there is no need to attempt to tell the story here in brief, for what reader does not know at least the outlines? It is indeed a great story ; did not an American physicist write on the occasion of J. J.'s seventieth birthday : " There is scarcely a physicist in America who has not been a pupil of Sir Joseph or else a pupil of one of his pupils "? Lord Rayleigh tells it well, with a clear appraisal of the genius of the Director and the skill and devotion of his collaborators. Mathematics is banned from the account. There are no formulae ; the author's technical mastery is such that difficult problems of physics, those problems tackled and solved in the Cavendish, are made plain to the general reader without them. But if we try to peer below

the surface, it is clear that mathematics is there, underlying the whole; J. J. was not Second Wrangler for nothing. Perhaps if he had been more of a physicist and less of a mathematician in the Cambridge of 1884 he would not at that time have become Cavendish Professor. Certainly he never underrated the importance of mathematics. We may quote from some remarks of his made at a joint meeting of the Mathematical Association and the Public School Science Masters' Association in 1910 (*Gazette*, V, p. 270): "At the present time there is a tendency to teach a kind of emasculated physics—a physics without mathematics. . . . There is a tendency to pander to the demands of the botanist and of the chemist who demand a kind of physics that would not give a headache to a caterpillar. . . . Mathematics and physics . . . ought never to be separated; they ought to march together almost from the very beginning."

Lord Rayleigh remarks that "with the possible exception of Bentley, no previous Master of Trinity could compete with Thomson in intellectual distinction"; one might ask whether any Chair can put forward a list of its first five occupants to compete with the Cavendish roll of Maxwell,* Rayleigh, Thomson, Rutherford and Bragg. T. A. A. B.

Infinite Series. By J. M. Hyslop. Pp. xi, 120. 5s. 1942. University Mathematical Texts. (Oliver and Boyd, Edinburgh)

This excellent little volume contains an extremely lucid account of the elementary properties of infinite series and products. It can be read, without difficulty, by anyone who possesses a knowledge of the elements of mathematical analysis, and should prove of great value to second- and third-year students.

The fundamental ideas on which the subsequent theory is based are discussed in the first two chapters. Four more chapters deal with series of positive terms, absolute and uniform convergence, and the work concludes with a treatment of series multiplication, infinite products and double series. Many instructive examples (so valuable to the reader and so often omitted in books of this sort) are worked out in the text, and in addition the author has provided a well-chosen collection to follow each of the nine chapters. With the exception of their application in the Cauchy-Maclaurin integral test, infinite integrals have been omitted.

By way of criticism, one feels that in connection with his treatment of Gamma Functions, the author might have included (at any rate as an example, if not in the text) the asymptotic formula $\Gamma(x+n) \sim n^x\Gamma(n)$, the proof of which presents no difficulty, and the importance of which it is difficult to overestimate. Another point which could have been clarified to advantage is the distinction between Fourier series and power series in regard to the extent of their intervals of uniform convergence. It only becomes apparent, in one example, that uniform convergence of a Fourier series in any closed sub-interval of the interval $0 < \theta < 2\pi$, together with convergence at the end points, does not imply uniform convergence in the closed interval $0 \leqslant \theta \leqslant 2\pi$, a marked contrast to the behaviour of power series in accordance with Abel's theorem.

In conclusion it seems fitting to congratulate all those who contributed to the preparation of this volume on the *entire* absence of errors in the answers to the examples. J. H. P.

* One very tiny point. It has become the fashion to give Clerk Maxwell a hyphen; this is done, for example, in two of the volumes of the new *Oxford History of England*, as it is in the present book. Is there any authority for this?

Analytical Geometry of Three Dimensions. By W. H. McCrea. Pp. vi, 144. 5s. 1942. (Oliver and Boyd)

Professor McCrea's contribution to the series "University Mathematical Texts" is as remarkable an achievement as was the first volume *Determinants and Matrices* by Aitken. In the space of only 138 pages of text we are given a clear, and even detailed, account of the analytical geometry of real three-dimensional space, an account that has a breath of freshness about it from beginning to end.

It is assumed that the student will read the book under supervision, so that explanations, whilst perfectly clear, are compressed, the examples forming an essential part of the text. The reader is assumed to have a knowledge of Aitken's book, but in fact a knowledge of the mere elements of the theory of matrices and determinants is sufficient for a complete understanding of the development.

We are told that the book is based upon a short course of lectures to first-year Honours students at the Queen's University of Belfast. The chapter headings read as follows : I. Coordinate system : directions ; II. Planes and Lines ; III. Sphere ; IV. Homogeneous coordinates—Points at infinity ; V. General equation of the second degree ; VI. Quadric in cartesian coordinates ; standard forms ; VII. Intersection of quadrics : systems of quadrics ; and there is a note on abstract geometry, and an index. The chapter headings alone indicate that, outside Belfast, *third*-year Honours students may profitably browse in these pastures. In fact, unless the war has changed everything, Chapter V is "outside the syllabus" of a London External Honours degree!

Cutting short these profitless reflections, it is good to see that among the many excellences of this book, such as fine paper and printing, and some masterly diagrams, there are included detailed instructions for making models of the various quadric surfaces. One wonders whether every school and university in the country has such geometrical models, and if not, why not? Their educational value is, without doubt, considerable, and Professor McCrea gives a lead to all teachers of geometry when he recommends that "some little time" should be devoted to model-making.

My one grouse at the book is that I wish it were bigger. Perhaps one day Professor McCrea will expand his present pocket-volume to one of true Salmon size. In the meantime we can all gratefully test our teeth on his present offering.

D. Pedoe.

Practical Calculations for the A.T.C. By T. H. Ward Hill. Pp. 75, +5 (answers). 2s. 1942. (Harrap)

This consists of those chapters from the same author's *Practical Mathematics for the A.T.C.* which are necessary for Part I of the Proficiency Examination, the whole of the trigonometry and some of the algebra and geometry being omitted. No alterations have been made, so that the book is still short in easy algebraical exercises for those who have done no algebra at school. The only additions are the stamp of official approval in the shape of a foreword by Air-Commodore Chamier, and an interesting set of mental exercises involving a little aeronautical general knowledge. The layout is attractively clear.

A. P. R.

Aircraft Calculations. By S. A. Walling and J. C. Hill. Pp. 131, +12 (tables), +16 (answers). 3s. 1942. (Cambridge)

This is a rearranged version of the same authors' *Aircraft Mathematics*, the first section of the book comprising the matter essential for the Part I of the Proficiency Examination, while further algebra, logarithms and trigonometry make up the second section. The many errors in the original

book have been removed, but it seems odd to find algebraical factors in the essential first section and the use of formulae in the optional second section. It is astonishingly hard for some writers to break away from the traditional order and content of elementary algebra. A. P. R.

Vital Arithmetic and Geometry. By R. S. Williamson. Pp. 96. 1s. 6d. Teacher's book. Pp. 96. 3s. 6d. 1942. (Cambridge University Press)

The author states that the book provides " a first course in Mathematics for Senior or Preparatory Schools ".

The arithmetic exercises cover the four rules for numbers, money and measures, and the use of simple fractions, with revision, test and project exercises. The questions are brief and the calculations are all with small numbers, and the degree of difficulty (or rather simplicity) throughout the exercises is constant.

A page taken at random is headed " Parts and Bills (£ s. d. $\div$) " and has as typical examples: one-eighth ($\frac{1}{8}$) of 17s.; price of 1 frock if 3 cost £5 12s. 6d.; if 4 lb. beef cost 5s. 4d. on 2nd Sept., 5 lb. beef cost 7s. 6d. on 13th Sept., Mary wants to know whether the price of beef has changed.

The chapters on geometry are exercises in the use of ruler, set-square and compasses, with a brief study of perpendicular lines, parallel lines, rectangles, triangles and circles, as they enter into simple design, mensuration, scale drawing and graphs. The ideas are *very simple*, with no general discussion of angles.

When judgment is based on experience of secondary school children, it must be tentative when applied to elementary school children, but the arithmetic material seems to me to be that normally covered in the Junior School from 7–11 years, and therefore, though the author does not say so, the book would be quite suitable for use in such schools. The average child would probably find the work comfortably within his range, though quicker children would need some supplementary material with harder numbers to satisfy their sense of adventure and power. Here the geometrical work could be valuable, for the ideas, though very simple, would be a sound preparation for a more formal course.

As far as the Senior (or Preparatory) school children are concerned, it appears to me that an A stream would find no stimulus in such simple material. With the B and C classes, however, the basic knowledge of arithmetical processes is often confused and shaky, or worse, and this book would provide very suitable material for such children. There is an entire absence of the intricate problems which delight the mentally agile, but are torture to the slow, and yet there are exercises calling for a little ingenuity to solve them. For such children the geometrical work would also be excellent, since the few simple ideas which are taught are those needed for work in Art, Geography and Domestic Science.

Recently much discussion and thought have been given to the reasons for teaching the subjects of the curriculum and the relation of each to the others, to the child's experience and future needs. This has led to books on " Citizenship Arithmetic ", many of which fail to achieve the high hopes set on them because the immaturity of the child cannot grasp many of the adult conceptions. The author makes much wider claims for his book than mere training in the tools of intelligent citizenship. He says: " The work has a human background . . . which draws upon that wider world which education is opening for him. . . . Capacities usually untouched by Mathematics, *e.g.* human sympathy, aesthetic feeling, humour, can find play, and his whole mental and emotional life should find adequate expression. The pupil can thus obtain wide experience of life." To carry out these somewhat ambitious

intentions there are a fair number of illustrations (of no great artistic merit) of, and references to, foreign countries, historical scenes, shops, furniture, etc. Such material, however admirable, cannot give a child "wide experience of life". That can only be obtained by living, but it would be fair to say that, intelligently used, these things can give to the child glimpses of a wider world lying outside his experience, and some idea that number and figures belong to reality and not to abstractness.

"Intelligently used" brings us to the "Notes and Suggestions" for the teacher, which give detailed notes on each exercise, dealing with material and method. It would doubtless provide the best basis for teaching if every teacher had time to gather stimulating material from wide sources on every subject, but that is impossible. These notes supply such ideas and information, but their value is much decreased by the impression that the author appears to trust no teacher other than himself. He is afraid that the teacher may use his material in some way other than the intended one, or that a single opportunity of spreading "culture" may be lost. This has turned the notes from a storehouse of interesting information into a course on teaching technique which will infuriate the abler teacher and scarcely help the weaker. If the study of elementary arithmetic and geometry is to fulfil even 10 per cent. of the functions outlined for it by Mr. Williamson, then it will only do so if teachers have the love of all aspects of life and experience in themselves, and this cannot be given by Notes, Instructions and Advice, however admirable and detailed. D. E. S.

Navigation for Air-crews. I. By J. E. C. Gliddon and E. C. Hedges. Pp. vi, 56, 8. 2s. 1942. (University of London Press)

This little book should be useful for revision purposes to those prospective pilots or navigators who have to take an examination in navigation, but it is hardly the book to put into the hands of a beginner. On opening this book for the first time, the reader might think that navigation consisted of a series of definitions and rules, and his interest might immediately evaporate. The examples to be worked in many cases amount to filling in blank spaces in a table and have little relation to the experience of a pupil when he first takes to the air. Nowadays many air-crew aspirants have had at least one flight in an aircraft, and if they have used their time wisely they will have indulged in some elementary map-reading navigation. This branch will always be one of the most important and reliable methods for finding one's way in the air, and every beginner should know something of it before trying to master other methods. Give a boy a map and tell him he is flying from Salisbury to Luton on a course of 045° and at 10.20 hrs. he crosses a wide river at a right-angled bend, and he can be shown the meaning of course, track, drift, airspeed and groundspeed, without any conscious effort being made. The formal definitions can follow at a later stage, as also the limitations of the particular map he is using. Most of the essential ideas used in navigation can be assimilated in this way before they are crystallised into definitions.

Apart from this (in the reviewer's opinion) mistaken approach to the subject in a book intended to encourage and inspire beginners, the information is accurately presented and includes several features not commonly found in a book of this scope. Variation and deviation are explained in more detail than usual, and a short section on compass swinging gives practice in their application. Very few inaccuracies were noticed, and the type and general layout are good. A small plotting chart is incorporated in the book and gives an opportunity for the reader to apply some of the principles he has learnt. This little volume fulfils its purpose as an elementary text-book, but fails in its aim of being a source of inspiration to would-be pilots or navigators. K. R. I.

Six figure Trigonometrical Tables and Formulae. Pp. 55. 1s. 6d. 1942. (Ford Motor Company, Dagenham, Essex)

Mathematicians in general, as well as engineers, will be interested in this publication of the Trade School of the Ford Motor Company. It consists primarily of a table giving to six decimal places the six trigonometrical ratios for every minute of arc. The remainder of the book is made up of a collection of elementary trigonometrical formulae. An interesting feature is a long list of numerical values of multiples and other functions of π.

The booklet is of a handy pocket size and shape, and the paper and binding are of the durable type which remind one of the motor handbooks which we used to study conscientiously in better days—or did we? B. M. B.

FOR SALE.

BOOKS RECEIVED FOR REVIEW.

H. F. Baker. *An introduction to plane geometry.* Pp. viii, 382. 18s. 1943. (Cambridge)

J. E. C. Gliddon and E. C. Hedges. *Navigation for air-crews. I.* Pp. vi, 56, 8. 2s. 1942. (University of London Press)

T. H. Ward Hill. *Practical calculations for the A.T.C.* Being chapters from *Practical mathematics for the A.T.C.* Pp. 80. 1942. (Harrap)

A. Hinckley. *Map projections by practical construction.* Pp. 84. 3s. 1942. (Philip)

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